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REDUCING THE ERROR OF GEOID UNDULATION COMPUTATIONS BY MODIFYING STOKES' FUNCTION

by

Christopher Jekeli

The Ohio State University
Department of Geodetic Science
1958 Neil Avenue
Columbus, Ohio 43210

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Abstract

The truncation theory as it pertains to the calculation of geoid undulations based on Stokes' integral, but from limited gravity data, is reexamined. Specifically, the improved procedures of Molodenskii et al. (1962) are shown through numerical investigations to yield substantially smaller errors than the conventional method that is often applied in practice. In this improved method, as well as in a simpler alternative (Meissl, 1971b) to the conventional approach, the Stokes' kernel is suitably modified in order to accelerate the rate of convergence of the error series. These modified methods, however, effect a reduction in the error only if a set of low-degree potential harmonic coefficients is utilized in the computation. Consider, for example, the situation in which gravity anomalies are given in a cap of radius 10° and the GEM 9 (20,20) potential field is used. Then, typically, the error in the computed undulation (aside from the spherical approximation and errors in the gravity anomaly data) according to the conventional truncation theory is 1.09 m; with Meissl's modification it reduces to 0.41 m, while Molodenskii's improved method gives 0.46 m. A further alteration of Molodenskii's method is developed and yields an RMS error of 0.33 m. These values reflect the effect of the truncation, as well as the errors in the GEM 9 harmonic coefficients. The considerable improvement, suggested by these results, of the modified methods over the conventional procedure is verified with actual gravity anomaly data in two oceanic regions, where the GEOS-3 altimeter geoid serves as the basis for comparison. The optimal method of truncation, investigated by Colombo (1977), is extremely ill-conditioned. It is shown that with no corresponding regularization, this procedure is inapplicable.

Foreword

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1. Introduction

The practical application of Stokes' solution to the geodetic boundary-value problem (Stokes' integral formula) has been studied extensively. The principal difficulty with this solution in practice is the formal requirement of continuous gravity data covering the entire geoid (approximated by a sphere). The lack of global coverage has led to the natural approximation whereby the integration is limited to a spherical cap ("truncation" of the integral). However, the data within a cap do not always constitute the total input, since much of the gravity information of the remaining exterior areas is often conveniently derived from a number of low-degree (say, up to degree 20) harmonic constituents which are determined, for example, by satellite methods.

The theory of this combination of satellite and terrestrial data to obtain the disturbing potential (or geoid undulation) originates with M. S. Molodenskii (Molodenskii, 1958; Molodenskii et al., 1962). The derivation of Molodenskii's basic theory has also been carried out by Heiskanen and Moritz (1967), and their exposition is the most familiar. However, Molodenskii (1958) (see also Molodenskii et al., 1962) extended and refined his theory leading to significant reductions in the truncation error (presuming a knowledge of a set of potential harmonic coefficients). In the recent literature, several authors have become aware of potential reductions in the error (Meissl, 1971b; Dickson, 1979), but Molodenskii must be credited with the most rigorous and comprehensive treatment. The applications of this theory, including further possible improvements, have been studied by Hsu Houtze (Hsu Houtze and Zhu Zhuowen, 1979).

It is the purpose of this report to bring to light Molodenskii's theory which seems to have gone largely unnoticed by practicing geodesists and to test its applicability with the current data available. In this respect, the gravity data inside the cap are assumed to be error free and sufficiently dense so that numerical integration errors can be neglected; on the other hand, errors in the harmonic coefficients of the gravity potential are allowed and treated accordingly.

Meissl (1971b) has suggested a very simple modification (which has also been used by Ostach (1970), see also Hsu Houtze and Zhu Zhuowen, 1979) to Stokes' function, effecting a substantial reduction in the truncation error if a higher-degree reference field is given. Both Meissl's approach and Molodenskii's original ideas rely on identical principles, but Molodenskii specifically builds on the premise that the geoid undulation computation should utilize potential harmonic coefficients. Meissl does not mention the use of harmonic coefficients and in applications his modification is quite useless in their absence. Nevertheless, since Meissl presents considerable mathematical insight into the truncation of Stokes' integral, his treatment will also be discussed in detail. The investigations would not be complete without mentioning the work of Wong and Gore (1969) whose method of introducing harmonic components is, however, somewhat arbitrary. Further tests of their

method are conducted in this report. Finally, the entirely different approach to the truncation theory of Colombo (1977) merits investigation. Colombo's solution, although yielding promising results, is only approximate because of the numerical instability of the problem posed by him. A rigorous solution is presented here, which however does not bypass the instability.

Most of the relevant derivations of Molodenskii, Meissl, and Wong and Gore are reproduced in the following sections, the only intention being to make this report, to some degree, self-contained. It is attempted to maintain a simple notation. Unfortunately, since a variety of kernels and modifications of kernels will enter the discussion, the symbolism must deviate occasionally from the usual notation found in the literature. The following will be adhered to:

Δg	geoidal gravity anomaly
Δg_n	nth surface harmonic function of Δg
$\Delta\hat{g}_n$	estimate of Δg_n based on known harmonic coefficients (which may be subject to error)
$\delta(\Delta g_n)$	the error in $\Delta\hat{g}_n$ such that $\Delta g_n = \Delta\hat{g}_n + \delta(\Delta g_n)$
c_n	degree variance of Δg_n
δc_n	degree variance of $\delta(\Delta g_n)$
N	geoid undulation given by Stokes' integral
$\hat{N}$	undulation computed by integrating over a cap, and possibly using a number of harmonic coefficients
δN	the error in $\hat{N}$, such that $N = \hat{N} + \delta N$
$\overline{\delta N}$	the RMS (root mean square, global average) of δN
$\overline{\delta N}'$	the RMS error, specifically when a number of harmonic coefficients are known
$S(y)$	Stokes' function, $y = \cos\psi$
$S^{*}(y)$	$S(y)$ minus its harmonics up to degree m
$\tilde{S}_{\bar{n}}(y)$	the first $\bar{n}+1$ harmonics of S when the latter is expanded in the interval $[-1, y_0]$, $y_0 < 1$
$K(y)$	the kernel used for the integration over the cap
$\Delta K(y)$	the error kernel of the integral representing δN
Q_n	a truncation coefficient (for any one of the kernels)

The individual modifications to the "classic" truncation theory are represented by subscripts, the subscript 1 referring to the classic theory. A subscript 2 denotes Meissl's (1971b) modification, a 3 refers to the method of Wong and Gore (1969), and an M signifies Molodenskii's (1958) modification. Other notations are explained in the text.

2. The Basic Problem

If N denotes the geoid undulation with respect to the mean earth ellipsoid, and Δg is the geoidal gravity anomaly, then, in spherical approximation, Stokes' integral reads (see Heiskanen and Moritz, 1967, pp. 92-94):

$$N(\theta,\lambda) = \frac{R}{4\pi\gamma}\iint_{\sigma} S(\cos\psi)\,\Delta g(\theta',\lambda')\,d\sigma \tag{1}$$

R is the radius of the sphere which approximates the geoid; γ is a mean value of normal gravity on this sphere; θ,λ are spherical coordinates; ψ is the central angle between points (θ,λ) and (θ',λ'); and σ denotes the unit sphere. Neither N nor Δg has zero- and first-degree harmonics under the assumption that the mass of the reference ellipsoid equals the earth's mass, that the normal potential on the ellipsoid equals the gravity potential on the geoid, and that the ellipsoid's center coincides with the earth's center of mass. The kernel $S(\cos\psi)$, known as Stokes' function is expandable in series form as

$$S(y) = \sum_{n=2}^{\infty} \frac{2n+1}{2}\, t_n\, P_n(y) \tag{2}$$

where $\cos\psi$ and its abbreviation

$$y = \cos\psi \tag{3}$$

will be used interchangeably throughout all derivations. The coefficients

$$t_n = \int_{-1}^{1} S(y)\, P_n(y)\, dy\,, \qquad n \geq 0$$

or

$$t_n = \frac{2}{n-1}\,, \qquad n \geq 2; \qquad t_0 = t_1 = 0 \tag{4}$$

are the Fourier coefficients of $S(y)$ (i.e. when expanded in terms of the (orthogonal) Legendre polynomials $P(y)$). Figure 1 shows $S(\cos\psi)$ as a function of ψ.

If gravity data are not available or sufficiently accurate over the entire globe, then the integration in (1) may be limited to a spherical cap σ centered at the computation point. This procedure is associated with an error; specifically

$$\delta N_1 = \frac{R}{4\pi\gamma}\iint_{\sigma} S(\cos\psi)\,\Delta g\, d\sigma - \frac{R}{4\pi\gamma}\iint_{\sigma_c} S(\cos\psi)\,\Delta g\, d\sigma \tag{5}$$

$$= \frac{R}{4\pi\gamma}\iint_{\sigma} \Delta K_1\,(\cos\psi)\,\Delta g\, d\sigma \tag{6}$$

(Technically, this is the negative of the error, but most authors seem to prefer the formulation in (5).) The error kernel ΔK_1 in this case is (see Figure 2)

Figure 1. Stokes' Function S(cos ψ)

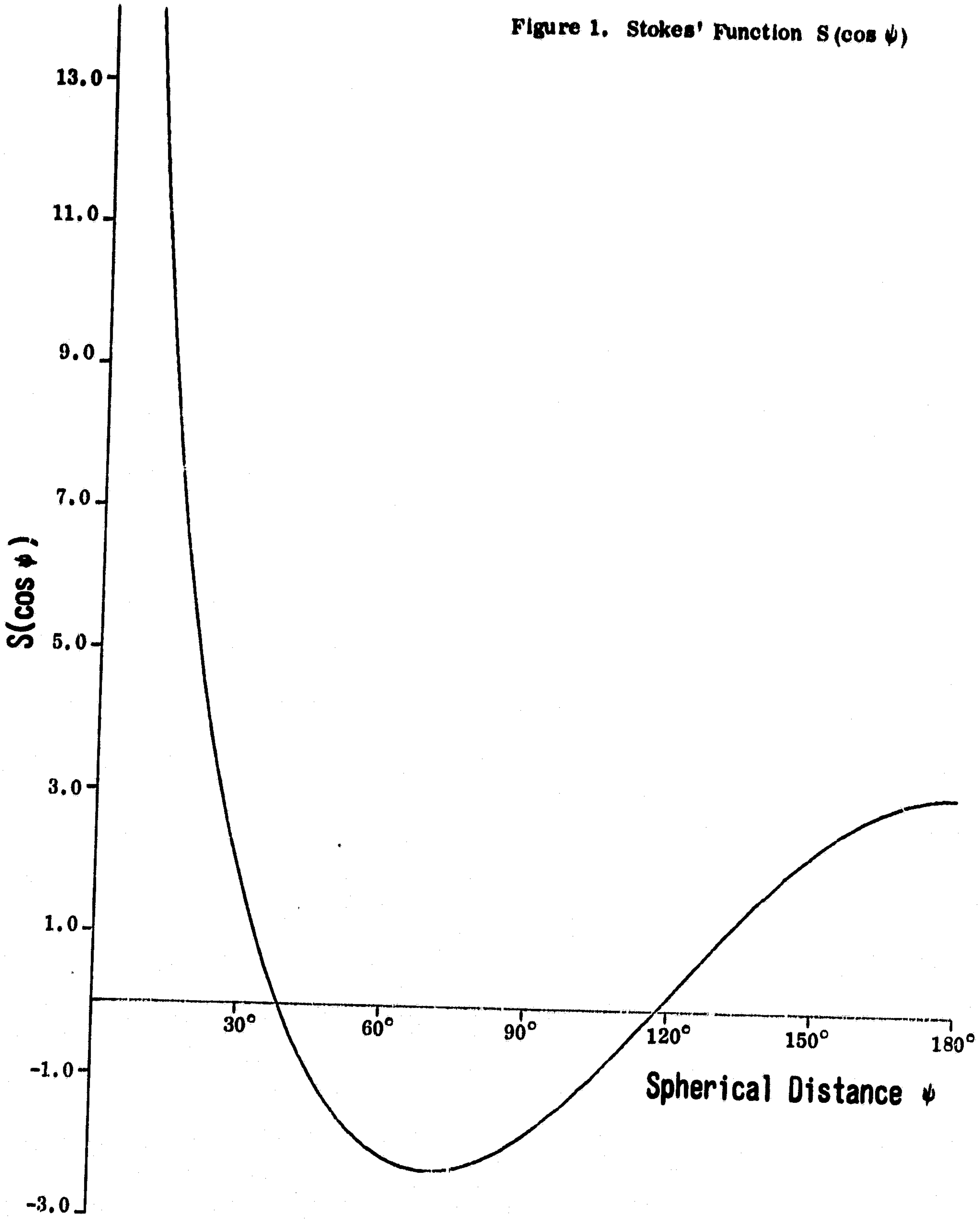
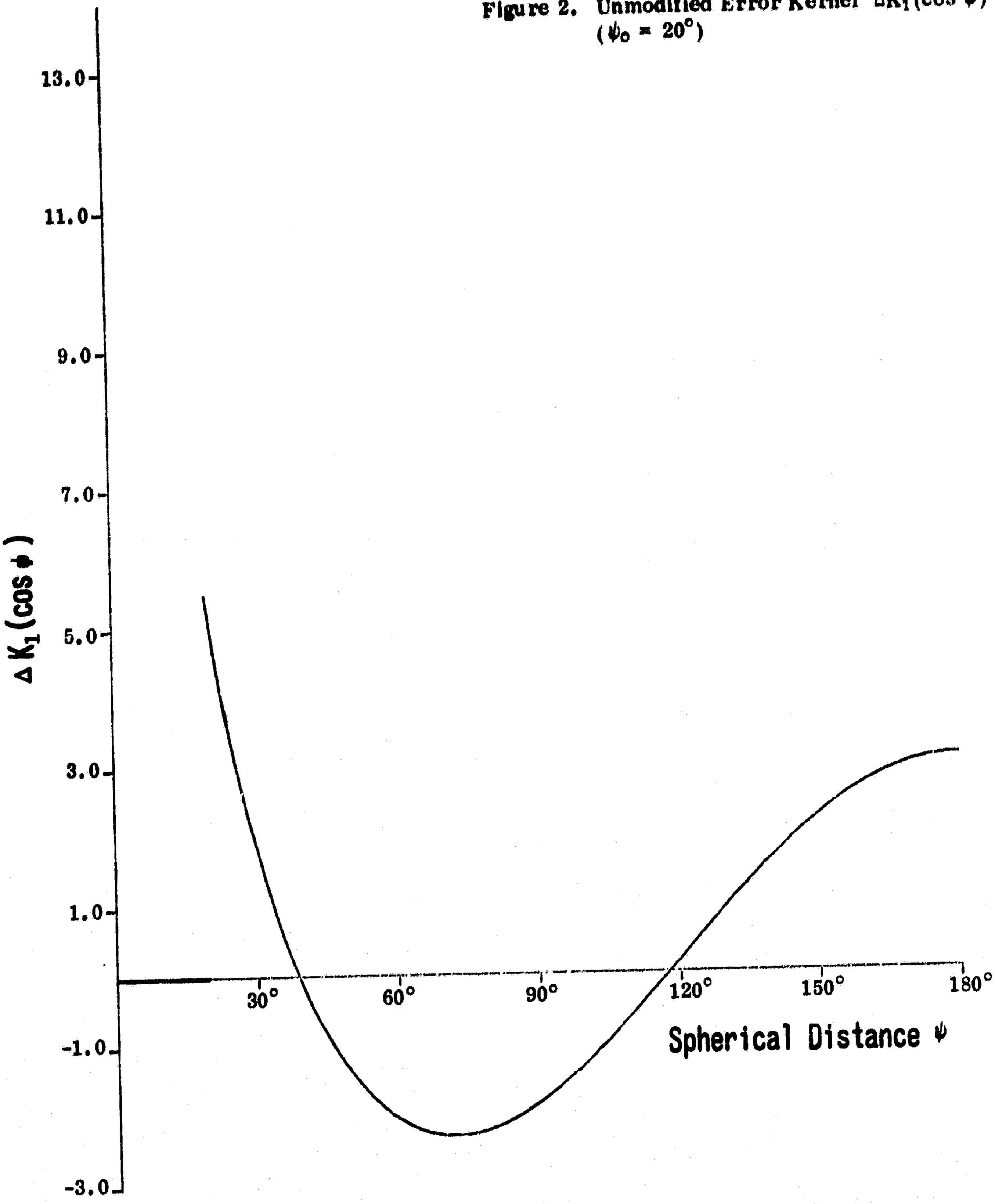


Figure 2. Unmodified Error Kernel $\Delta K_1(\cos\psi)$ ($\psi_0 = 20^\circ$)



$$\Delta K_1(\cos\psi) = \begin{cases} 0, & 0 < \psi \le \psi_0 \\ S(\cos\psi), & \psi_0 < \psi \le \pi \end{cases} \tag{7}$$

where ψ_0 is the radius (spherical distance) of the cap σ_c.

Expanding the surface function Δg into spherical harmonic components Δg_{nm}, we may write briefly

$$\Delta g(\theta,\lambda) = \sum_{n,m} \Delta g_{nm}(\theta,\lambda) \tag{8}$$

The spherical harmonic functions Δg_{nm} are eigenfunctions of the integral operator of the type

$$\iint_\sigma (\cdot)\, S(\cos\psi)\, d\sigma \tag{9}$$

That is, this operator applied to Δg_{nm} returns Δg_{nm} multiplied by a constant, or eigenvalue (depending in this case only on the degree n):

$$\iint_\sigma \Delta g_{nm}(\theta',\lambda')\, S(\cos\psi)\, d\sigma = 2\pi t_n \Delta g_{nm}(\theta,\lambda); \quad n,m \ge 0 \tag{10}$$

This useful fact is readily established as follows. Explicitly, we have

$$\Delta g_{nm}(\theta,\lambda) = A_{nm}\overline{R_{nm}}(\theta,\lambda) + B_{nm}\overline{S_{nm}}(\theta,\lambda); \quad n,m \ge 0 \tag{11}$$

where $\overline{R_{nm}}$, $\overline{S_{nm}}$ are fully normalized spherical harmonic functions, and A_{nm}, B_{nm} are constant coefficients (see Heiskanen and Moritz, 1967, p. 31). The series expansion of S (equation (2)) is now substituted into (10), and using the addition theorem for $P_n(\cos\psi)$ (ibid, p. 33):

$$P_n(\cos\psi) = \frac{1}{2n+1}\sum_{m=0}^{n} [\overline{R_{nm}}(\theta,\lambda)\,\overline{R_{nm}}(\theta',\lambda') + \overline{S_{nm}}(\theta,\lambda)\,\overline{S_{nm}}(\theta',\lambda')]; \quad n \ge 0 \tag{12}$$

we obtain

$$\iint_\sigma \Delta g_{nm}(\theta',\lambda')\, S(\cos\psi)\, d\sigma = \sum_{r=2}^{\infty} \frac{2r+1}{2} t_r \frac{1}{2r+1} \iint_\sigma [A_{nm}\overline{R_{nm}}(\theta',\lambda') + B_{nm}\overline{S_{nm}}(\theta',\lambda')] \cdot$$

$$\cdot \sum_{q=0}^{r} [R_{rq}(\theta,\lambda)\, R_{rq}(\theta',\lambda') + S_{rq}(\theta,\lambda)\, S_{rq}(\theta',\lambda')]\, d\sigma\ ; \quad n,m \ge 0 \tag{13}$$

Invoking now the orthogonality of spherical harmonic functions, we get

$$\iint_\sigma S(\cos\psi)\Delta g_{nm}(\theta',\lambda')\, d\sigma = \tfrac{1}{2} t_n \left[A_{nm}\overline{R}_{nm}(\theta,\lambda)\iint_\sigma \overline{R}^2_{nm}(\theta',\lambda')\, d\sigma + B_{nm}\overline{S}_{nm}(\theta,\lambda)\iint_\sigma \overline{S}^2_{nm}(\theta',\lambda')\, d\sigma\right]$$

$$= \tfrac{1}{2} t_n \cdot 4\pi\, \Delta g_{nm}(\theta,\lambda) = 2\pi t_n \Delta g_{nm}(\theta,\lambda); \quad n,m \ge 0 \tag{14}$$

thus proving equation (10). This result holds in general for any kernel function depending only on ψ and will be used repeatedly. It conveniently provides for the immediate conversion of an integral over the sphere σ to a series of harmonics.

Therefore, the expansion of the error δN_1 into spherical harmonics is simply achieved by first expanding the error kernel (equation (7)):

$$\Delta K_1(y) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_{1n} P_n(y) \tag{15}$$

where $y = \cos\psi$ and the Q_{1n} are the Fourier coefficients of this expansion:

$$Q_{1n} = \int_{-1}^{1} \Delta K_1(y)\, P_n(y)\, dy \tag{16}$$

$$= \int_{-1}^{y_0} S(y)\, P_n(y)\, dy \tag{17}$$

Here, $y_0 = \cos\psi_0$. As above, we note that the constants $2\pi Q_{1n}$ are the eigenvalues of the integral operator in (6). Then substituting (8) into (6) yields

$$\delta N_1 = \frac{R}{4\pi\gamma} \sum_{n,m} \iint_{\sigma} \Delta K_1(\cos\psi)\, \Delta g_{nm}\, d\sigma$$

$$= \frac{R}{2\gamma} \sum_{n,m} Q_{1n} \Delta g_{nm}$$

or

$$\delta N_1 = \frac{R}{2\gamma} \sum_{n=2}^{\infty} Q_{1n} \Delta g_n \tag{18}$$

where

$$\Delta g_n = \sum_{m=0}^{n} \Delta g_{nm} \tag{19}$$

(Recall that by assumption $\Delta g_0 = 0$; also $\Delta g_1 = 0$.) The coefficients Q_{1n}, being functions of ψ_0, are generally known as Molodenskii's truncation coefficients (or functions). The rigorous evaluation of Q_{1n} according to (17) has been undertaken, e.g., by Paul (1973) who developed an accurate recursion formula (see also Hagiwara, 1976). Hsu Houtze and Zhu Zhuowen (1979) have derived approximate formulas for Q_{1n} when n is large; see also Ganeko (1977). Thus, the error in the undulation committed by neglecting gravity anomalies Δg outside the cap σ_i can be computed according to (18). Usually, for numerical studies such as in this report, the error is estimated by a global average of δN_1, i.e. the root mean square (RMS) value $\overline{\delta N_1}$:

$$(\overline{\delta N_1})^2 = \frac{R^2}{4\gamma^2} \sum_{n=2}^{\infty} Q_{1n}^2 c_n \tag{20}$$

where the quantities c_n are degree variances of Δg, given on the geoid (sphere of radius R). The derivation of (20) may be found in Heiskanen and Moritz (1967, p. 261). The degree variances c_n, not known to a very high degree, are often evaluated according to a model (see Tscherning and Rapp, 1974).

Formula (20) has been applied frequently in practice, for example, by Rapp and Rummel (1975) who used a slightly modified version, taking into account a higher-degree reference field (see section 3); also Wong and Gore (1969) and Fell (1978) have investigated its characteristics. In each instance, it was observed that the RMS error, paradoxically, does not always decrease as a larger field of anomalies is integrated (cap size is increased) (see Figure 5). Mathematically, this phenomenon is easily explained. Recall that Q_{1n} is a function of ψ_0, and if $y_0 = \cos\psi_0$, then

$$\frac{d}{d\psi_0} Q_{1n} = -\sin\psi_0 \frac{d}{dy_0} Q_{1n}, \qquad n > 0, \qquad \psi_0 \neq 0 \tag{21}$$

Assuming that the function Q_{1n} has a minimum value, it must occur when $\frac{d}{dy_0} Q_{1n} = 0$. From (17)

$$\frac{d}{dy_0} Q_{1n} = S(y_0)\, P_n(y_0), \qquad n > 0, \qquad y_0 \neq 1 \tag{22}$$

This is zero for all n if $S(y_0) = 0$. Consequently, if the RMS error $\overline{\delta N}$, (equation (20)) has minima, they are at the zeros of $S(\cos\psi)$, namely at $\psi_0 = 38^\circ\!.962073$ and $\psi_0 = 117^\circ\!.66153$, as seen in Figure 5. This strong dependence of the error on the kernel is the motivation for the discussions of Meissl (1971b) and is the basis for any modification to the kernel, the objective being to reduce the error.

In summary then, the geoid undulation, formally the result of an integration over the entire sphere, is computed by limiting the integration to a spherical cap, thus incurring an error. The following sections review the modifications to the error kernel as proposed by various authors. The error kernel is, in theory, modified only in its definition over the cap, requiring of course a corresponding change in the kernel of the integral of gravity anomalies in order to preserve the integrity of the original Stokes' formula (1).

A strong analogy exists between this idea and the concept of window functions in spectral analysis. By restricting the integration to a cap, we "see" merely a portion of the global gravity information, as if through a window. Thus, a change in the kernel, i.e. in the weights of the integration (applying a different window function; e.g. giving less weight to the Δg near the edge of the cap) can, under certain circumstances, lead to a better approximation of the true undulation.

3. Meissl's Modification

The following formulas, which form part of Meissl's (1971b) derivation of a recurrence formula for the Q_{1n} coefficients, hold for any kernel function. Here, they are specialized to Stokes' function.

Let f, g be two functions, twice differentiable and square integrable; then the derivation of the following analogy to Green's second identity may be found in Meissl (1971a, p. 43):

$$\iint_B (f\nabla^2 g - g\nabla^2 f)\, d\sigma = \int_{\delta B} (f\vec{\nabla} g - g\vec{\nabla} f)\cdot\vec{\upsilon}\, d\,\delta B \tag{23}$$

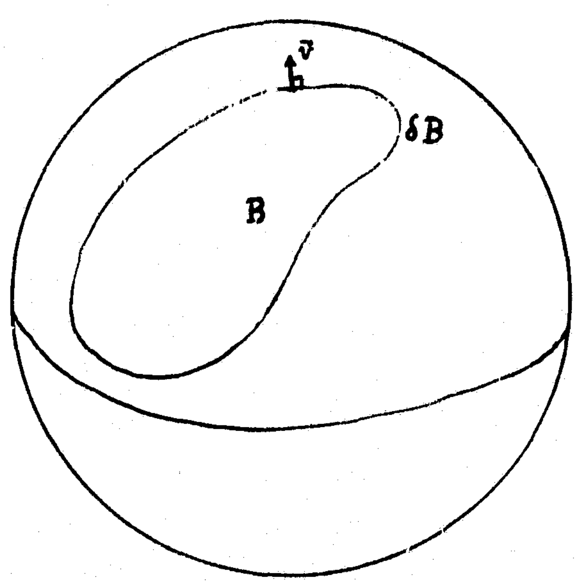
where B is any sub-area of the unit sphere σ. δB is the boundary of B, and $\vec{\upsilon}$ is a unit vector tangent to B and normal to δB such that it is directed away from B (see Figure 3). ∇^2 and $\vec{\nabla}$ are the Laplacian and gradient operators, respectively, both formulated in the spherical coordinate system r, θ, λ, holding r constant (=1). For example,

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \lambda^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial \theta} \tag{24}$$

which if r = constant = 1 becomes

$$\nabla^2(f\,|_{r=1}) = \frac{\partial^2 f}{\partial \theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \lambda^2} + \cot\theta\,\frac{\partial f}{\partial \theta} \tag{25}$$

Figure 3. Region B on Unit Sphere σ



The reformulation of equation (17) for Q_{1n} using Green's second identity as expressed in (23) will enable us to study the asymptotic behavior of these coefficients. Without loss in generality, the pole of the spherical coordinate system may be rotated to coincide with the point at which the undulation is computed. Then ψ is the co-latitude and both $S(\cos\psi)$ and $P_n(\cos\psi)$ are independent of the second variable, say the azimuth α. In equation (23), we may identify $f(\psi,\alpha)$ with $S(\cos\psi)$, $\nabla^2 g(\psi,\alpha)$ with $P_n(\cos\psi)$, B with $\sigma - \sigma_c$, and δB with β which denotes the circle bounding the cap $\sigma - \sigma_c$. Then (23) becomes

$$\int_{\alpha=0}^{2\pi}\int_{\psi=\psi_0}^{\pi} [S(\cos\psi)\, P_n(\cos\psi) - \nabla^{-2} P_n(\cos\psi)\, \nabla^2 S(\cos\psi)] \sin\psi \, d\psi \, d\alpha$$
$$= \int_{\beta} [S(\cos\psi)\, \vec{\nabla}(\nabla^{-2} P_n(\cos\psi)) - \nabla^{-2} P_n(\cos\psi)\, \vec{\nabla} S(\cos\psi)] \cdot \vec{\upsilon}\, d\beta \tag{26}$$

where ∇^{-2} is the inverse Laplacian operator, i.e. $\nabla^{-2}(\nabla^2 f) = f = \nabla^2(\nabla^{-2} f)$. Now with $y = \cos\psi$, (25) is transformed into (with $\psi \equiv \theta$, $\lambda \equiv \alpha$)

$$\nabla^2(f|_{r=1}) = (1-y^2)\frac{\partial^2 f}{\partial y^2} - 2y\frac{\partial f}{\partial y} + \frac{1}{1-y^2}\frac{\partial^2 f}{\partial \alpha^2}$$
$$= \frac{\partial}{\partial y}\left((1-y^2)\frac{\partial f}{\partial y}\right) + \frac{1}{1-y^2}\frac{\partial^2 f}{\partial \alpha^2} \tag{27}$$

Since $S(\cos\psi)$ and $P_n(\cos\psi)$ are independent of α, the second term in (27) vanishes. The following are familiar formulas for $P_n(y)$ (Hobson, 1965, pp. 32-33):

$$(n+1)P_{n+1}(y) + n\,P_{n-1}(y) = (2n+1)\,y\,P_n(y), \quad n \geq 1 \tag{28}$$

$$(1-y^2)\,P_n'(y) = n(P_{n-1}(y) - y\,P_n(y)), \quad n \geq 1 \tag{29}$$

$$y\,P_n'(y) - P_{n-1}'(y) = n\,P_n(y), \quad n \geq 1 \tag{30}$$

$$(2n+1)\,P_n(y) = P_{n+1}'(y) - P_{n-1}'(y), \quad n \geq 1 \tag{31}$$

(the primes denote differentiation with respect to the argument y). Using (29), (30) and (27), it is easily verified that

$$\nabla^2 P_n(y) = -n(n+1)\,P_n(y), \quad n \geq 0 \tag{32}$$

so that
$$\nabla^{-2} P_n(y) = \frac{-1}{n(n+1)}\,P_n(y), \quad n \geq 1 \tag{33}$$

The boundary β is the circle $\psi = \psi_0$ with (linear) radius $\sin\psi_0$; and hence the differential arc element is $d\beta = \sin\psi_0\, d\alpha$, where $0 \leq \alpha \leq 2\pi$. Also, the unit vector $\vec{\upsilon}$ is in the direction of decreasing ψ, implying that $\vec{\nabla} f \cdot \vec{\upsilon}$, being the directional derivative of f in the direction of $\vec{\upsilon}$, is simply

$$\vec{\nabla} f \cdot \vec{\upsilon} = \frac{-\partial f}{\partial \psi} \quad \text{and} \quad -\frac{\partial f(y)}{\partial \psi} = +\sin\psi \; f'(y) \tag{34}$$

With these considerations, (26) reduces to

$$2\pi\int_{-1}^{y_0}[S(y)P_n(y) + \frac{1}{n(n+1)}P_n(y)\nabla^2 S(y)]\,dy = \frac{-1}{n(n+1)}\int_0^{2\pi}[S(y_0)P_n'(y_0) - P_n(y_0)S'(y_0)](1-y_0^2)d\alpha$$
$$= \frac{-2\pi(1-y_0^2)}{n(n+1)}[S(y_0)P_n'(y_0) - P_n(y_0)S'(y_0)], \quad n \geq 1 \tag{35}$$

where the operator ∇^2 is given by (27). Our original aim to reformulate the expression for Q_{1n} is achieved by recalling (17):

$$Q_{1n} = \frac{-1}{n(n+1)}\int_{-1}^{y_0}P_n(y)\nabla^2 S(y)\,dy - \frac{(1-y_0^2)}{n(n+1)}[S(y_0)P_n'(y_0) - P_n(y_0)S'(y_0)], \; n \geq 1 \tag{36}$$

This, in essence, is equation (B.50) of Meissl (1971b).

Now if $n \to \infty$, as Meissl notes, the first and third terms of (36) approach zero as $1/n^2$ while due to $P_n'(y)$ such rapid convergence is not guaranteed for the second term (see (29)). Therefore, the condition $S(y_0) = 0$ accelerates the approach of the coefficients Q_{1n} to zero and the convergence rate of the series for δN_1 (equation (20)) is increased. This observation is the basis for Meissl's strategy to reduce the error $\overline{\delta N}_1$.

From equations (1) and (5), we may write

$$N = \iint_{\sigma_c}[S(\cos\psi) - S_0]\Delta g\,d\sigma + \delta N_1 + \iint_{\sigma_c} S_0 \Delta g\,d\sigma \tag{37}$$

where $S_0 = S(\cos\psi_0)$ and where the error is now

$$\delta N_2 = \delta N_1 + \iint_{\sigma_c} S_0 \Delta g\,d\sigma = \iint_{\sigma}\Delta K_2(\cos\psi)\,\Delta g\,d\sigma \tag{38}$$

with the error kernel (see Figure 4)

$$\Delta K_2(\cos\psi) = \begin{cases} S_0, & 0 < \psi \leq \psi_0 \\ S(\cos\psi), & \psi_0 < \psi \leq \pi \end{cases} \tag{39}$$

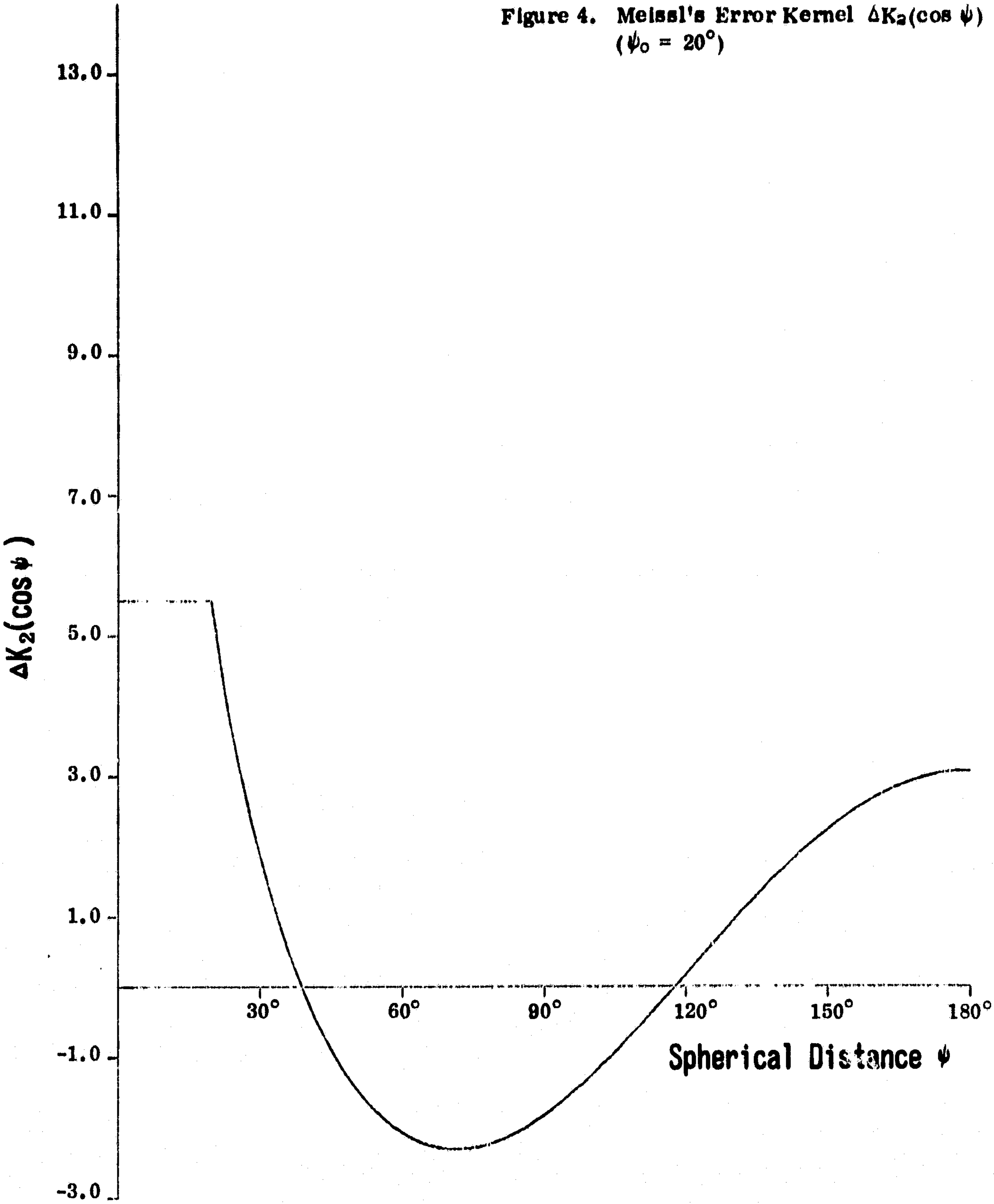
$\Delta K_2(\cos\psi)$ can be expanded in a series of Legendre polynomials:

$$\Delta K_2(y) = \sum_{n=0}^{\infty}\frac{2n+1}{2}Q_{2n}P_n(y) \tag{40}$$

where the Q_{2n} are the Fourier coefficients:

$$Q_{2n} = \int_{-1}^{1}\Delta K_2(y)P_n(y)\,dy, \quad n \geq 0 \tag{41}$$
$$= S_0\int_{-1}^{1}P_n(y)\,dy + \int_{-1}^{y_0}[S(y) - S_0]P_n(y)\,dy$$

Figure 4. Meissl's Error Kernel $\Delta K_2(\cos\psi)$
$(\psi_0 = 20°)$



$$Q_{2n} = 0 + \int_{-1}^{y_0} [S(y) - S_0]\, P_n(y)\, dy, \quad n \geq 1 \tag{42}$$

which follows by the orthogonality of Legendre polynomials. Proceeding as before, the constants $2\pi Q_{2n}$ are the eigenvalues of the integral operator in (38), and the RMS error $\overline{\delta N_2}$ is given by

$$\overline{\delta N_2} = \frac{R}{2\gamma}\left[\sum_{n=2}^{\infty} Q_{2n}^{\,2}\, c_n\right]^{\frac{1}{2}} \tag{43}$$

Applying the kernel $K_2(y) = S(y) - S_0$ to equation (35) and considering (42), we obtain

$$Q_{2n} = \frac{-1}{n(n+1)} \int_{-1}^{y_0} P_n(y)\nabla^2 S(y)\, dy - \frac{(1-y_0^2)}{n(n+1)}[0 - P_n(y)S'(y_0)], \quad n \geq 1 \tag{44}$$

since $\nabla^2(S(y)-S_0) = \nabla^2 S(y)$, $S(y_0) - S_0 = 0$, and $\frac{d}{dy}[S(y) - S_0]\big|_{y=y_0} = S'(y_0)$. Therefore as $n \to \infty$, the coefficients Q_{2n}, lacking the second term in (36), approach zero faster than the Q_{1n}. A comparison of the functions $\Delta K_1(y)$ and $\Delta K_2(y)$ (Figures 2 and 4) makes this equally evident. $\Delta K_2(y)$ is a continuous function, while $\Delta K_1(y)$ has a jump-discontinuity at $y = y_0$. Since each of the partial sums of the expansion (15) of ΔK_1 is a continuous function, the series converges rather hesitantly to the discontinuous function. Particularly in the neighborhood of y_0 (if y_0 is not a zero of $S(y)$), the partial sums exhibit considerable oscillations in their attempt to accommodate the discontinuity (Gibbs phenomenon - see any textbook on Fourier series) and in the limit converge to $S_0/2$ at $y = y_0$. It is therefore also obvious that the series for the continuous function ΔK_2 enjoys a much improved rate of convergence. This rate is further enhanced by removing also the discontinuity of the first (and higher) derivative at $y = y_0$; however, this is not pursued here (see Meissl, 1971b, p. 52; cf. Hsu Houtze and Zhu Zhuowen, 1979). It must be noted that the convergence rate is not a critical issue in the computation of the RMS error with today's computers. The upper limit of the sum (equation (20) or (43)) is easily taken to 2000 or 3000 which is normally sufficient for at least millimeter accuracy (somewhat less for $\psi_0 = 0°$) in the approximation of the error series by a finite sum.

Although the coefficients Q_{2n} converge to zero more rapidly than Q_{1n}, this does not ensure an RMS error $\overline{\delta N_2}$ that is smaller than $\overline{\delta N_1}$ (see equations (20) and (43)). In fact, for cap radii ψ_0 less than 40°, the error $\overline{\delta N_2}$ exceeds $\overline{\delta N_1}$, as clearly seen in Figure 5. Table 1 lists values of Q_{2n} versus Q_{1n} for $\psi_0 = 10°$ verifying the accelerated convergence of Q_{2n} to zero, but also showing that more "power" is shifted to the low-degree harmonics of the error kernel ΔK_2, whence the larger error. Evidently, the error $\overline{\delta N_2}$ is significantly smaller than $\overline{\delta N_1}$ for a value of ψ_0 such as 65°; but considering the current state of world-wide gravity data, caps with radii larger than 20° or 30° are usually deficient in coverage that is both dense and accurate.

Figure 5. RMS error in N due to truncation at cap radius ψ_o, using the usual Stokes' function (curve I) and Meissl's modification (curve II).

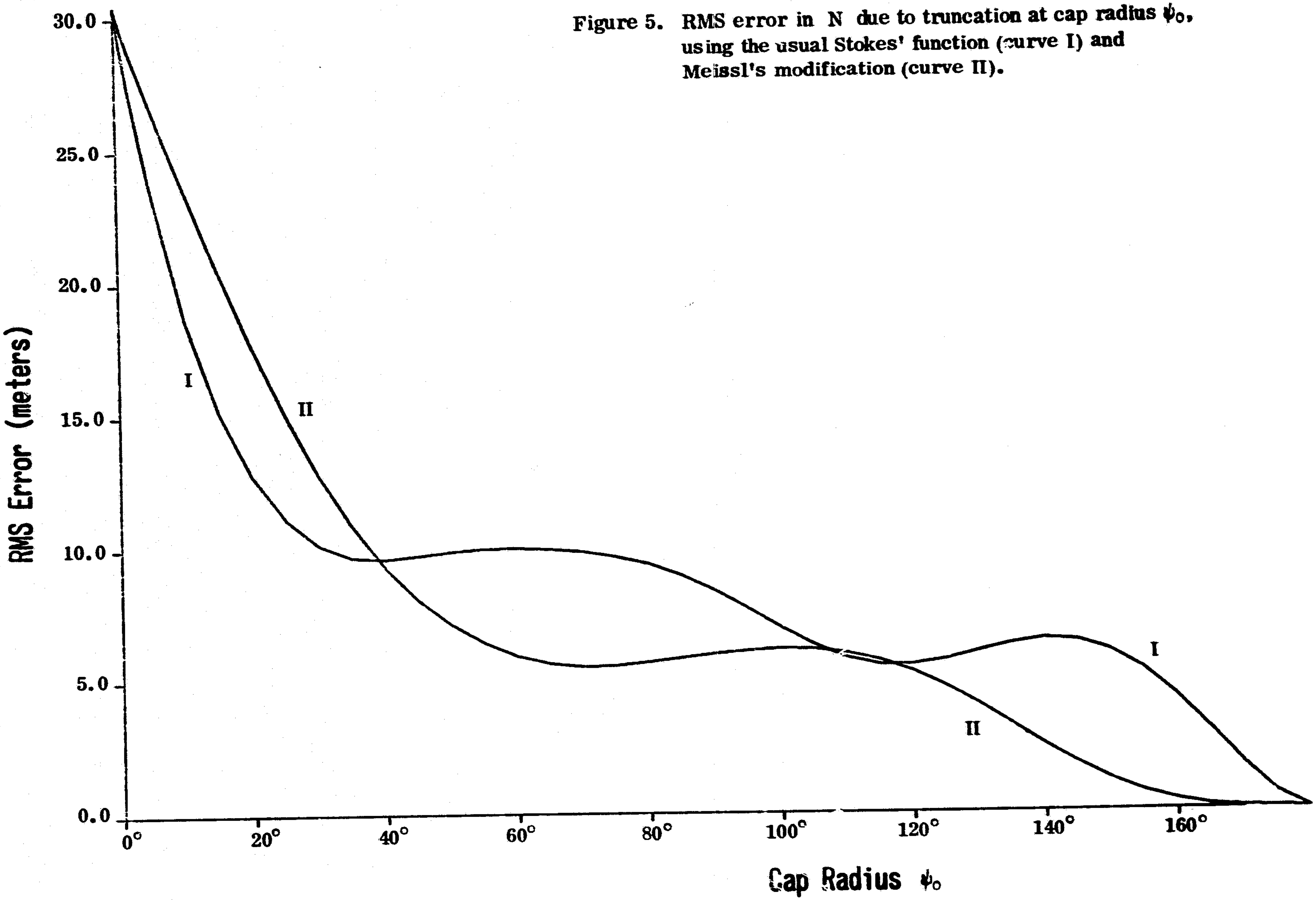


Table 1. Truncation Coefficients of the Error Kernels ΔK_1, ΔK_2, ΔK_3 ($\psi_0 = 10°$)

n	Q_{1n}	Q_{2n}	Q_{3n} (m = 20)
0	-.414	-.201	.034
1	-.411	-.201	-.051
2	1.593	1.801	.032
3	.599	.802	.030
4	.274	.471	.027
5	.118	.307	.025
6	.030	.210	.021
7	-.023	.147	.017
8	-.056	.103	.013
9	-.076	.072	8.36×10^{-3}
10	-.086	.049	3.46×10^{-3}
15	-.073	-3.65×10^{-3}	-.023
20	-.025	-.012	-.047
25	.013	-7.80×10^{-3}	.020
30	.026	-1.65×10^{-3}	4.38×10^{-4}
50	-.013	-1.33×10^{-4}	2.05×10^{-3}
100	3.89×10^{-5}	-1.54×10^{-4}	-3.79×10^{-4}
150	-7.[illegible]6 $\times 10^{-4}$	9.89×10^{-5}	-1.78×10^{-8}
200	-5.91×10^{-4}	-4.73×10^{-5}	1.41×10^{-4}
300	-8.77×10^{-4}	3.51×10^{-6}	1.22×10^{-4}
500	4.10×10^{-4}	8.48×10^{-7}	-6.00×10^{-5}
1000	1.27×10^{-4}	-4.61×10^{-7}	-1.78×10^{-5}
1500	2.72×10^{-5}	-3.13×10^{-7}	-3.55×10^{-6}

Meissl's modification of the kernel therefore has practical applicability if the low-degree harmonic components of the gravity field are known (not necessarily perfectly); the corresponding terms then do not contribute to the truncation error. Let m denote the maximum degree of the available potential harmonic coefficients. Then one may compute according to the conventional approach

$$\hat{N}_1 = \frac{R}{4\pi\gamma} \iint_{\sigma_c} S(\cos\psi)\,\Delta g\,d\sigma + \frac{R}{2\gamma} \sum_{n=2}^{m} Q_{1n}\,\hat{\Delta g}_n \tag{45}$$

with an error

$$\delta N_1' = \frac{R}{2\gamma} \sum_{n=2}^{m} Q_{1n}\,\delta(\Delta g_n) + \frac{R}{2\gamma} \sum_{n=m+1}^{\infty} Q_{1n}\,\Delta g_n \tag{46}$$

(assuming no errors in the data of the cap σ_c), or by removing the discontinuity of the truncated kernel:

$$\hat{N}_2 = \frac{R}{4\pi\gamma} \iint_{\sigma_c} [S(\cos\psi) - S_0]\,\Delta g\, d\sigma + \frac{R}{2\gamma} \sum_{n=2}^{m} Q_{2n}\, \Delta\hat{g}_n \tag{47}$$

with an error

$$\delta N_2' = \frac{R}{2\gamma} \sum_{n=2}^{m} Q_{2n}\, \delta(\Delta g_n) + \frac{R}{2\gamma} \sum_{n=m+1}^{\infty} Q_{2n}\, \Delta g_n \tag{48}$$

The quantities $\Delta\hat{g}_n$ are the harmonic functions computed from the coefficients. The corresponding error is

$$\delta(\Delta g_n) = \Delta g_n - \Delta\hat{g}_n\,, \quad 2 \le n \le m \tag{49}$$

The RMS estimates of $\delta N_1'$ and $\delta N_2'$ are given by

$$\overline{\delta N_1}' = \frac{R}{2\gamma}\left[\sum_{n=2}^{m} Q_{1n}^2\, \delta c_n + \sum_{n=m+1}^{\infty} Q_{1n}^2\, c_n\right]^{\frac{1}{2}} \tag{50}$$

and

$$\overline{\delta N_2}' = \frac{R}{2\gamma}\left[\sum_{n=2}^{m} Q_{2n}^2\, \delta c_n + \sum_{n=m+1}^{\infty} Q_{2n}^2\, c_n\right]^{\frac{1}{2}} \tag{51}$$

where δc_n is the degree variance of the errors in the harmonic components of degree n. The validity of (50) and (51) follows immediately under the hypothesis of zero correlation between the errors in the coefficients and the harmonics above degree m.

Wong and Gore (1969) adopted a conceptually different method (with no claims to reduce the error) in which the first $m - 1$ harmonic components are subtracted from the kernel (see also Fell (1978)). Recalling (2), let

$$S^m(\cos\psi) = \sum_{n=m+1}^{\infty} \frac{2n+1}{2}\, t_n\, P_n(\cos\psi) \tag{52}$$

then the computation of the undulation according to

$$\hat{N} = \frac{R}{4\pi\gamma} \iint_{\sigma_c} S^m(\cos\psi)\,\Delta g\, d\sigma \tag{53}$$

is associated with an error

$$\begin{aligned}
\delta N_3 &= \frac{R}{4\pi\gamma}\left[\iint_{\sigma} S(\cos\psi)\,\Delta g\, d\sigma - \iint_{\sigma_c} S^m(\cos\psi)\,\Delta g\, d\sigma\right] \\
&= \frac{R}{4\pi\gamma}\left[\iint_{\sigma} S(\cos\psi) \sum_{n=2}^{m} \Delta g_n d\sigma + \iint_{\sigma} S(\cos\psi)\,\Delta g^m\, d\sigma\, + \right. \\
&\qquad \left. + \iint_{\sigma-\sigma_c} S^m(\cos\psi)\,\Delta g\, d\sigma - \iint_{\sigma} S^m(\cos\psi)\,\Delta g\, d\sigma\right] \\
&= \frac{R}{4\pi\gamma}\left[\iint_{\sigma-\sigma_c} S^m(\cos\psi)\,\Delta g\, d\sigma + \iint_{\sigma} S(\cos\psi) \sum_{n=2}^{m} \Delta g_n\, d\sigma\, + \right. \\
&\qquad \left. + \iint_{\sigma} [S(\cos\psi) - S^m(\cos\psi)]\,\Delta g^m d\sigma - \iint_{\sigma} S^m(\cos\psi) \sum_{n=2}^{m} \Delta g_n d\sigma\right]
\end{aligned} \tag{54}$$

where $$\Delta g^{m} = \sum_{n=m+1}^{\infty} \Delta g_n \tag{55}$$

The last two terms in (54) vanish by the orthogonality of harmonic functions, since $S(\cos\psi) - S^{m}(\cos\psi)$ possesses no harmonics beyond degree m and S^{m} has none below degree $m+1$. Equation (54) can therefore be rewritten as

$$\delta N_3 = \frac{R}{4\pi\gamma}\left[\iint_{\sigma-\sigma_c} S^{m}(\cos\psi)\,\Delta g^{m} d\sigma + \iint_{\sigma-\sigma_c} S^{m}(\cos\psi) \sum_{n=2}^{m} \Delta g_n\, d\sigma + \iint_{\sigma} S(\cos\psi) \sum_{n=2}^{m} \Delta g_n\, d\sigma\right] \tag{56}$$

Given potential coefficients to degree m (subject to errors $\delta(\Delta g_n)$), the second and third terms of (56) may be included in the evaluation of the undulation. Hence we calculate

$$\hat{N}_3 = \frac{R}{4\pi\gamma}\iint_{\sigma_c} S^{m}(\cos\psi)\,\Delta g\, d\sigma + \frac{R}{2\gamma}\sum_{n=2}^{m}(Q_{3n} + t_n)\,\Delta\hat{g}_n \tag{57}$$

with an error

$$\delta N_3' = \frac{R}{2\gamma}\sum_{n=2}^{m}(Q_{3n} + t_n)\,\delta(\Delta g_n) + \frac{R}{2\gamma}\sum_{n=m+1}^{\infty} Q_{3n}\,\Delta g_n \tag{58}$$

where

$$Q_{3n} = \int_{-1}^{y_0} S^{m}(y)\, P_n(y)\, dy\,, \qquad n \geq 0 \tag{59}$$

are the Fourier coefficients of the expansion of the error kernel (cf. equation (56)):

$$\Delta K_3(\cos\psi) = \begin{cases} 0 & 0 < \psi \leq \psi_0 \\ S^{m}(\cos\psi)\,, & \psi_0 < \psi \leq \pi \end{cases} \tag{60}$$

and the coefficients t_n are given by (4). Numerical values of Q_{3n} for $\psi_0 = 10^\circ$ are listed in Table 1. The RMS error $\overline{\delta N_3'}$, from (58), is

$$\overline{\delta N_3'} = \frac{R}{2\gamma}\left[\sum_{n=2}^{m}(Q_{3n} + t_n)^2\,\delta c_n + \sum_{n=m+1}^{\infty} Q_{3n}^2\, c_n\right]^{\frac{1}{2}} \tag{61}$$

Note that ΔK_3, expanded in the interval $[-1,1]$ as a series of Legendre polynomials, contains all harmonics from degree zero to infinity. The constants $2\pi Q_{3n}$ are also the eigenvalues of the integral operator in (56), whence the results (57) and (58) follow easily (see section 2).

This section is concluded with a numerical comparison of the three alternative RMS errors in the undulation that are caused by the lack of complete and global gravity data, as well as the errors in the harmonic coefficients. Specifically, these error estimates are given by equations (50), (51), and (61). Equation (50) is the RMS error if Stokes' function $S(\cos\psi)$ is applied to the cap σ_c; equation (51) corresponds to the use of $S(\cos\psi) - S_0$ in σ_c; while (61) represents the error when $S^{m}(\cos\psi)$, as defined in (52), is used.

The gravity anomaly degree variances are modelled according to Tscherning and Rapp (1974):

$$c_n = \frac{425.28\,(n-1)}{(n-2)(n+24)}\, s^{n+2}\ \text{mgal}, \qquad n \geq 3 \tag{62}$$

where $s = .999617$; and the degeee variances δc_n are determined from

$$\delta c_n = \gamma^2 (n-1)^2\, \xi_n\,, \qquad n \geq 2 \tag{63}$$

where the ξ_n are the degree variances of the errors $\delta\bar{C}_{nm}$, $\delta\bar{S}_{nm}$ in fully normalized harmonic coefficients of degree n. The formula for ξ_n is

$$\xi_n = \sum_{m=0}^{n} [\,(\delta\bar{C}_{nm})^2 + (\delta\bar{S}_{nm})^2\,], \qquad n \geq 2 \tag{64}$$

The estimates of ξ_n, as listed in Table 2, were obtained by substituting into (64) the standard deviations of the coefficients of the GEM 9 (20,20) solution (Lerch et al., 1977).

Table 2. Estimates of Degree Variances of GEM 9 Potential Coefficient Errors

n	$\xi_n \times 10^{12}$	n	$\xi_n \times 10^{12}$
2	.000037	12	.004471
3	.000341	13	.006040
4	.000205	14	.005048
5	.000964	15	.006343
6	.000656	16	.005538
7	.002130	17	.006873
8	.001522	18	.007575
9	.003536	19	.006882
10	.002940	20	.007038
11	.005949		

The Molodenskii truncation coefficients Q_{1n} can be computed, for example, by the recursive formulas of Paul (1973). Then with relatively little additional effort the coefficients Q_{2n} are obtained from (36), (44), and (29) as

$$Q_{2n} = Q_{1n} + \frac{S_0}{n-1}\,[\,P_{n-1}(y_0) - y_0\,P_n(y_0)\,], \qquad n \geq 1 \tag{65}$$

where $S_0 = S(\cos\psi_0)$, $y_0 = \cos\psi_0$. Finally, the coefficients Q_{1n} can also be modified to yield the coefficients Q_{3n}. Substituting (52) into (59) and considering (2), we have

$$Q_{mn} = Q_{1n} - \sum_{r=2}^{m} \frac{2r+1}{2} t_r \int_{-1}^{y_0} P_r(y)\, P_n(y)\, dy, \qquad n \ge 0 \tag{66}$$

$$= Q_{1n} - \sum_{r=2}^{m} \frac{2r+1}{2} t_r\, e_{rn}$$

where

$$e_{rn} = \int_{-1}^{y_0} P_r(y)\, P_n(y)\, dy, \qquad r, n \ge 0 \tag{67}$$

Then Hobson (1965, p. 38) gives

$$e_{rn} = \frac{1}{(r-n)(r+n+1)} [n P_r(y_0) P_{n-1}(y_0) - r P_n(y_0) P_{r-1}(y_0) + y_0(r-n) P_n(y_0) P_r(y_0)], \tag{68}$$

$$r \neq n, \quad r, n > 0$$

Also

$$e_{r,0} = \frac{1}{2r+1} [P_{r+1}(y_0) - P_{r-1}(y_0)], \qquad r > 0 \tag{69}$$

$$e_{0,0} = 1 + y_0$$

$$e_{r,r} = \frac{1}{2r+1} [(2r-1) e_{r-1,r-1} + y_0(P_r^2(y_0) + P_{r-1}^2(y_0)) - 2 P_r(y_0) P_{r-1}(y_0)], \quad r > 0 \tag{70}$$

Equation (69) is obtained by inserting (31) into (67) and noting that $P_r(-1) = (-1)^r$. Equation (70) is derived in Appendix A. The values $R = 6371$ km and $\gamma = kM/R^2 = 982026.$ mgal (where $kM = 398601\ km^3/sec^2$ is the product of the gravitational constant and the earth's mass) were used in all computations.

Figure 6 shows the RMS errors for cap radii $\psi_0 = 0°, \ldots, 40°$ with the assumption of no errors in the harmonic coefficients up to degree $m = 20$; i.e., $\delta c_n = 0$, $n = 2, \ldots, 20$. Clearly exhibited is the substantial difference in magnitude between the $\overline{\delta N_1}'$ and $\overline{\delta N_2}'$ truncation errors, even for smaller caps. For example, at $\psi_0 = 10°$, $\overline{\delta N_1}' = .82$ m, $\overline{\delta N_2}' = .26$ m, and $\overline{\delta N_3}' = .82$ m. Both the unmodified error kernel and the kernel of Wong and Gore (1969) produce similar truncation errors; but the introduction of errors into the reference field causes the two corresponding RMS errors to diverge considerably, as seen in Figure 7. This characteristic is attributable to the coefficients t_n in equation (61) which are independent of the cap radius ψ_0; and therefore, $\overline{\delta N_2}'$ reflects primarily the large error due to the erroneous harmonic coefficients (e.g., an RMS contribution of 1.61 m for $\psi_0 = 20°$).

Of the three alternative methods, considered in this section, to compute undulations, Figure 7 evidently favors (for $\psi_0 \ge 4°$) the simple modification elaborated upon by Meissl (1971b). Furthermore, the feasibility of integrating caps of radius between 5° and 10°, and yet achieving half-meter accuracy (assuming a 20-degree reference field such as GEM 9 and neglecting integration and gravity data errors), is manifestly demonstrated. Figure 7 gives, for example at $\psi_0 = 10°$, $\overline{\delta N_1}' = 1.09$ m, $\overline{\delta N_2}' = 0.41$ m, and $\overline{\delta N_3}' = 1.67$ m. It may be noted, again, that the improved convergence rate of $\overline{\delta N_2}'$ does not produce a corresponding reduced magnitude for $\psi_0 < 4°$.

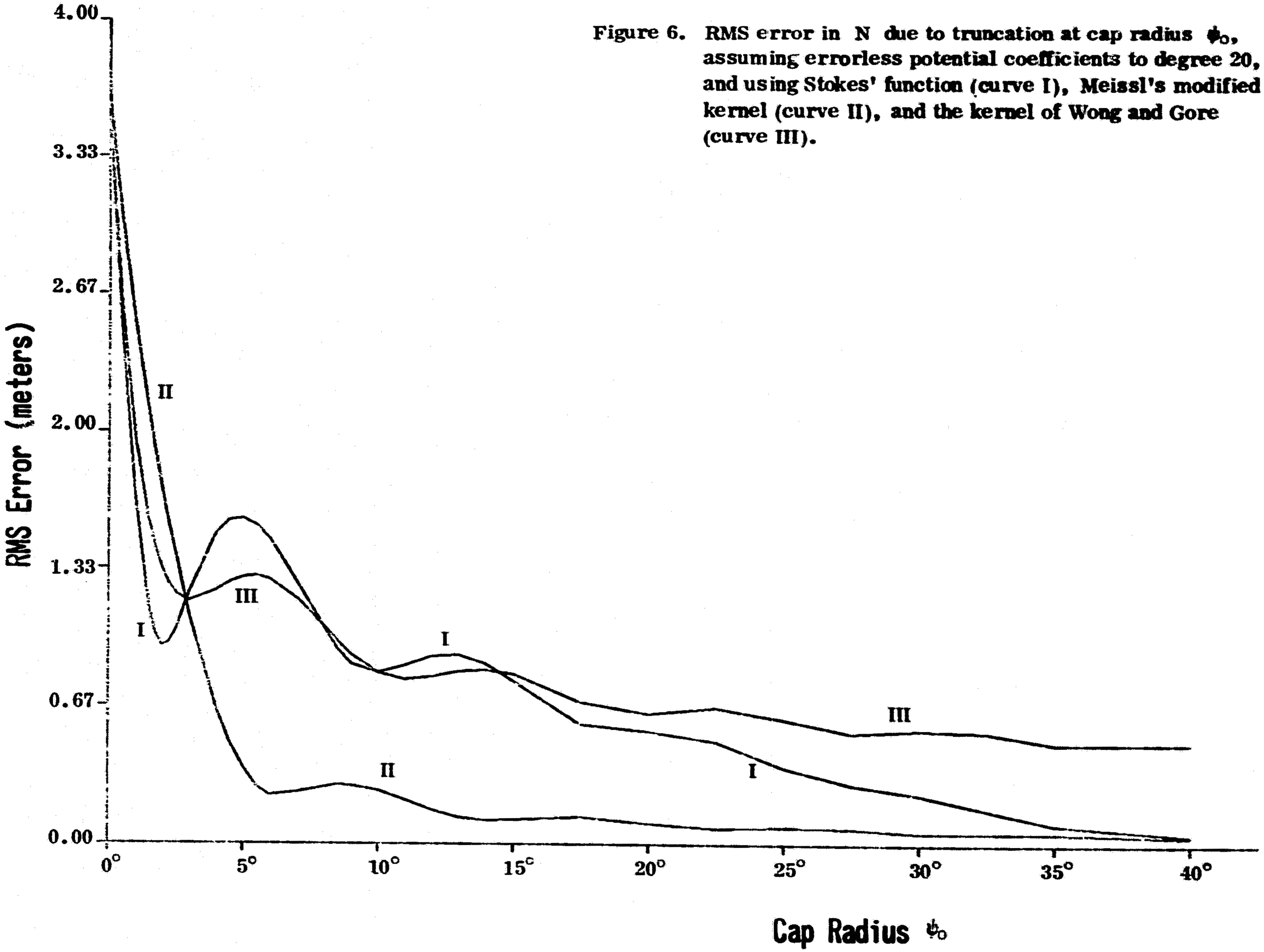


Figure 6. RMS error in N due to truncation at cap radius ψ_0, assuming errorless potential coefficients to degree 20, and using Stokes' function (curve I), Meissl's modified kernel (curve II), and the kernel of Wong and Gore (curve III).

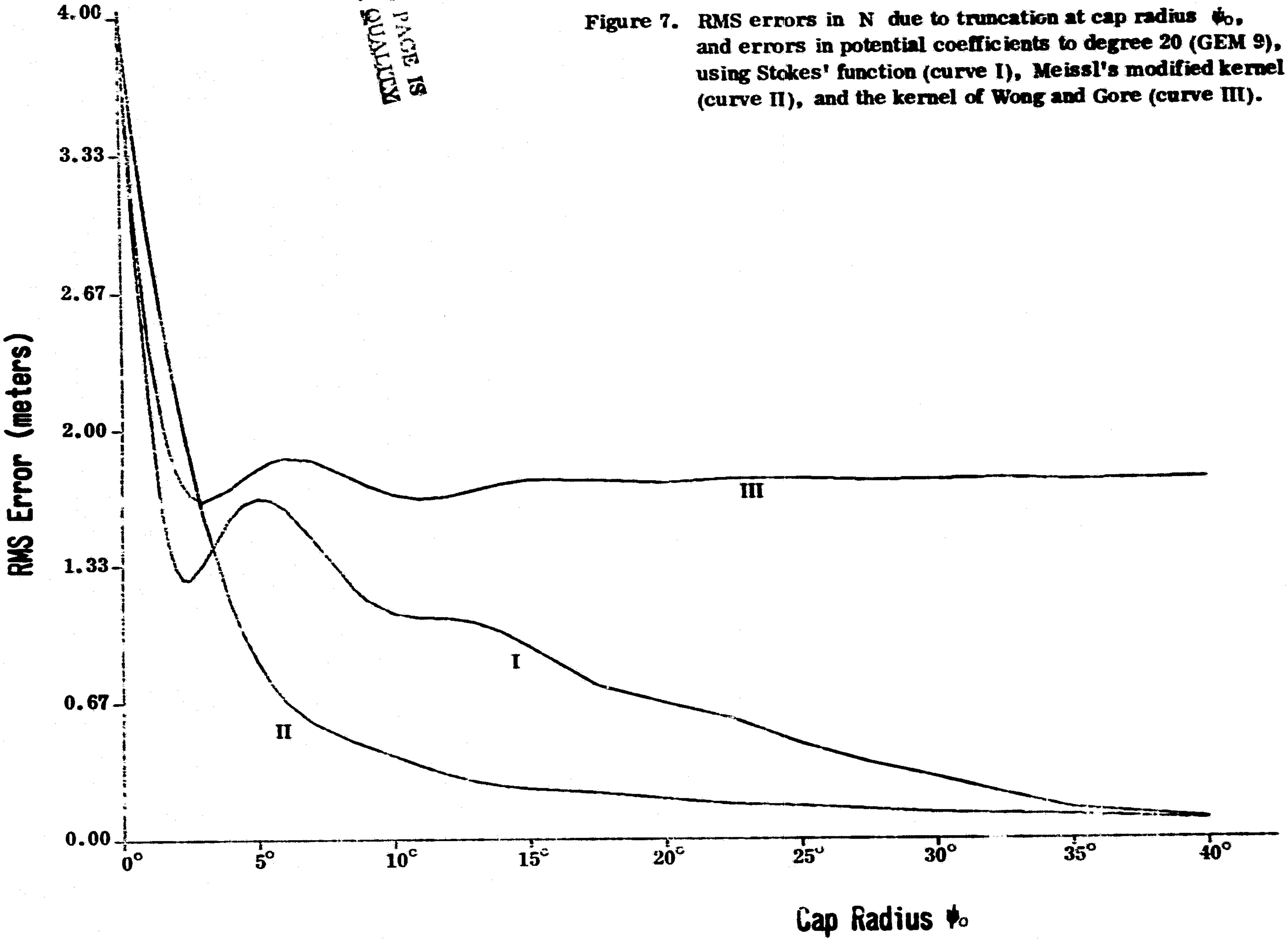


Figure 7. RMS errors in N due to truncation at cap radius ψ_0, and errors in potential coefficients to degree 20 (GEM 9), using Stokes' function (curve I), Meissl's modified kernel (curve II), and the kernel of Wong and Gore (curve III).

4. Molodenskii's Improvement

Molodenskii's approach to reduce the truncation error, originally (Molodenskii, 1958) devised as the determination of the most rapidly converging error series, adds sophistication to the concept of section 3 as more than a mere constant term is subtracted from Stokes' function. Recapitulating the problem, the error in the undulation derived from an integration of gravity anomalies over a spherical cap (radius ψ_0), using Stokes' function as the kernel, may be represented by an integral over the entire sphere where the error kernel is zero over the cap (see section 2). When this error is expanded in a series, the coefficients converge to zero slowly as a result of the kernel's discontinuity at $\psi = \psi_0$. However, if the error kernel were to be redesigned so as to eliminate not only its discontinuity, but also the discontinuities at ψ_0 of all its derivatives, then we should have the smoothest error kernel possible with the most rapidly converging error series. This modification of the error kernel amounts to an analytic continuation over the cap of the truncated Stokes' function (i.e., from $\psi = 0$ to $\psi = \psi_0$). But then we would encounter a dilemma since Stokes' function itself is analytic for $0 < \psi \leq \pi$, and the error kernel continued thus would coincide with Stokes' function exactly for $0 < \psi \leq \pi$. That is, an analytic function is determined uniquely by the values of it and all its derivatives at a single point - in our case, these values are already specified for all points in the interval $[\psi_0, \pi]$. Consequently, as easily recognized from equations (71) and (72) below, the kernel for the integration over the cap would be zero, nullifying the contribution of the gravity anomalies in σ_c.

A viable alternative (and this is how Molodenskii proceeded) is to approximate Stokes' function in the interval $[\psi_0, \pi]$ by a finite sum of Legendre polynomials. This new kernel enjoys all the properties of being analytic over the entire interval $[0, \pi]$, but an additional error is introduced since it does not coincide exactly with Stokes' function for $\psi_0 < \psi < \pi$. Specifically, let the undulation be computed according to

$$\hat{N} = \frac{R}{4\pi\gamma} \iint_{\sigma_c} K_M(\cos\psi)\, \Delta g\, d\sigma \tag{71}$$

where

$$K_M(\cos\psi) = S(\cos\psi) - \tilde{S}_{\bar{n}}(\cos\psi), \quad 0 < \psi \leq \psi_0 \tag{72}$$

and the function

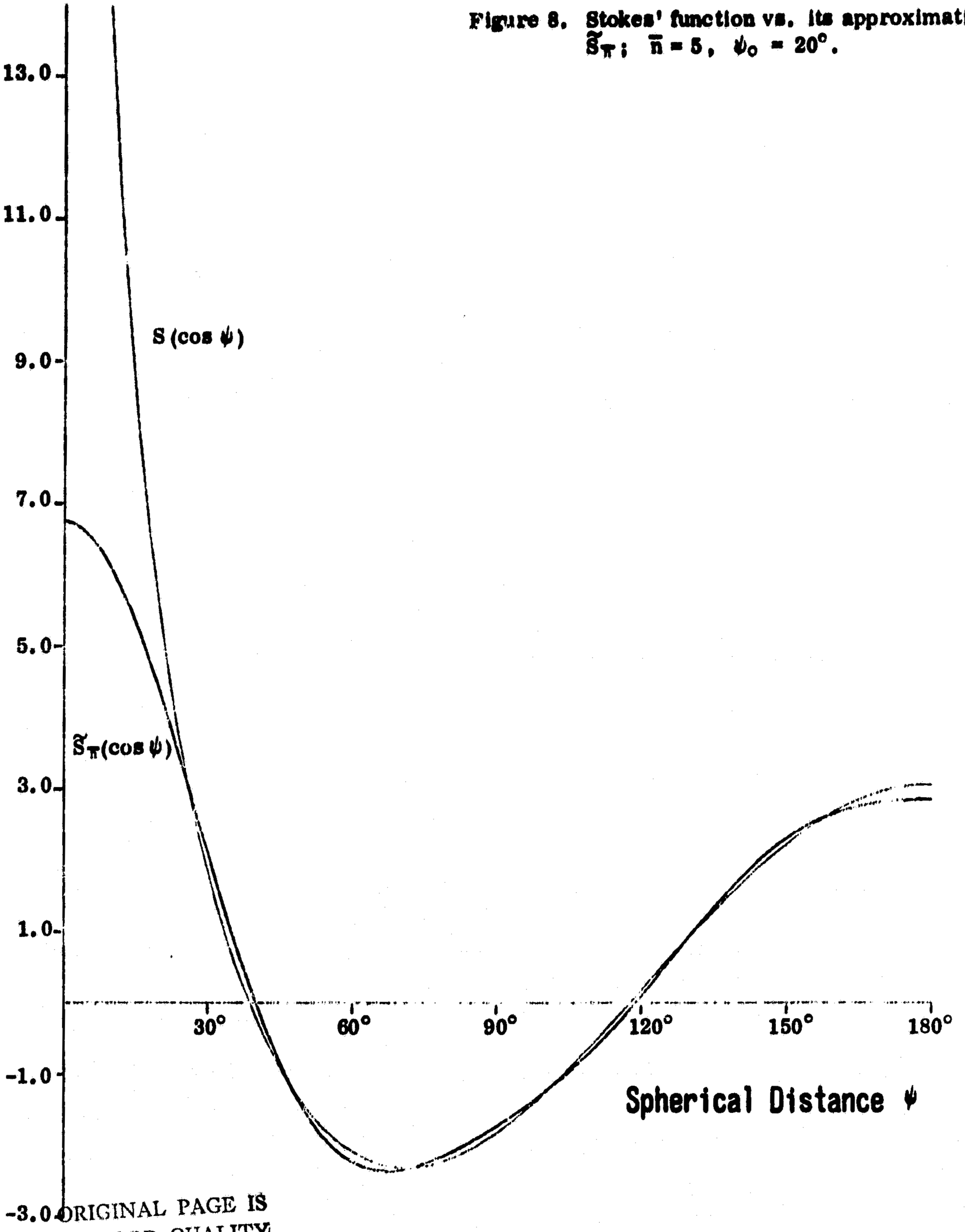
$$\tilde{S}_{\bar{n}}(y) = \sum_{n=0}^{\bar{n}} \frac{2n+1}{2}\, s_n P_n(y) \tag{73}$$

is to be defined by the coefficients s_n so as to render the "best" approximation to $S(y)$ for $\psi_0 \leq \psi \leq \pi$ (see Figure 8). This is not to be confused with the approach followed by Wong and Gore (1969) since $s_n \neq t_n$ - the concern here is only with the interval $[\psi_0, \pi]$ (see below for the determination of s_n). The error of $\hat{N}$ is

$$N - \hat{N} = \frac{R}{4\pi\gamma} \iint_{\sigma} S(\cos\psi)\Delta g\, d\sigma - \frac{R}{4\pi\gamma} \iint_{\sigma_c} [S(\cos\psi) - \tilde{S}_{\bar{n}}(\cos\psi)]\,\Delta g\, d\sigma$$

$$= \frac{R}{4\pi\gamma} \iint_{\sigma - \sigma_c} S(\cos\psi)\Delta g\, d\sigma + \frac{R}{4\pi\gamma} \iint_{\sigma_c} \tilde{S}_{\bar{n}}(\cos\psi)\Delta g\, d\sigma \tag{74}$$

Figure 8. Stokes' function vs. its approximation $\tilde{S}_{\pi}$; $\bar{n} = 5$, $\psi_0 = 20°$.



$$N - \hat{N} = \frac{R}{4\pi\gamma}\iint_{\sigma-\sigma_c}[S(\cos\psi) - \tilde{S}_{\bar{n}}(\cos\psi)]\Delta g\, d\sigma + \frac{R}{4\pi\gamma}\iint_{\sigma}\tilde{S}_{\bar{n}}(\cos\psi)\sum_{n=2}^{\bar{n}}\Delta g_n d\sigma \qquad (75)$$

The last equality follows by the orthogonality of harmonic functions, since $\tilde{S}_{\bar{n}}(y)$ contains no harmonics above degree $\bar{n}$. The first term in (75) represents the error due to the discrepancy between $S(y)$ and $\tilde{S}_{\bar{n}}(y)$ in $[\psi_0, \pi]$. (If $n \to \infty$, then this term is zero since $\tilde{S}_{\bar{n}}(y) \to S(y)$; but this is true for all y in $[-1, 1]$, so that also $\hat{N} \to 0$.) If potential coefficients to degree m are available (with possible errors), then we may set $\bar{n} = m$, thereby essentially eliminating the second term of the error (equation (75)). Thus, the undulation is computed according to

$$\hat{N}_M = \frac{R}{4\pi\gamma}\iint_{\sigma_c}[S(\cos\psi) - \tilde{S}_m(\cos\psi)]\Delta g\, d\sigma + \frac{R}{2\gamma}\sum_{n=2}^{m} s_n \hat{\Delta g}_n \qquad (76)$$

with an error

$$\delta N_M' = \frac{R}{4\pi\gamma}\iint_{\sigma-\sigma_c}[S(\cos\psi) - \tilde{S}_m(\cos\psi)]\Delta g\, d\sigma + \frac{R}{2\gamma}\sum_{n=2}^{m} s_n\, \delta(\Delta g_n) \qquad (77)$$

where, obviously by (73), the constants $2\pi s_n$ are also the eigenvalues of the second integral operator in (75).

Before proceeding to further simplify (77), $\tilde{S}_m$ must be determined explicitly. Recall that $\tilde{S}_m$, being a finite sum of polynomials, is to approximate S as closely as possible in the interval $[\psi_0, \pi]$. In the recent papers which explore this method of reducing the error (Dickson, 1979; Fell and Karaska, 1979; cf. Hsu Houtze and Zhu Zhuowen, 1979) the function $\tilde{S}_m$ is determined through a "least squares adjustment". More precisely, by quantifying the concept of closeness through the definition of a norm, a well known and powerful theorem of Fourier analysis yields the desired result. Let $f(x)$ be a function, square integrable in the interval $[-1, 1]$; then the norm of f may be defined as the square root of

$$\|f\|^2 = \int_{-1}^{1} f^2(x)\, dx \qquad (78)$$

Now change the variable y $(= \cos\psi)$ to

$$y = kx + k - 1; \qquad k = \cos^2\frac{\psi_0}{2} = \frac{1}{2}(1 + \cos\psi_0) \qquad (79)$$

As y ranges from -1 to y_0, x varies from -1 to 1. We note that while the Legendre polynomials $P_n(y)$ do not form an orthogonal basis in the interval $[-1, y_0]$, such a basis is formed by the polynomials $P_n(x)$. $\tilde{S}_m(y)$ given by (73) (with $\bar{n} = m$) is a polynomial of degree m and can equally well be expanded in terms of $P_n(x)$:

$$\tilde{S}_m(y) = \sum_{r=0}^{m} \frac{2r+1}{2} u_r P_r(x) \qquad (80)$$

where, formally, the Fourier coefficients u_r are given by

$$u_r = \int_{-1}^{1} \tilde{S}_s(kx+k-1)\, P_r(x)\, dx = \frac{1}{k}\int_{-1}^{y_0} \tilde{S}_s(y)\, P_r\left(\frac{y-k+1}{k}\right) dy\,; \quad k \neq 0,\ r \geq 0 \tag{81}$$

The "best" approximation of $\tilde{S}_s$ to S in the interval $[-1, y_0]$, or equivalently for $-1 \leq x \leq 1$, is then obtained by minimizing the norm of the difference:

$$\int_{-1}^{1} [S(y) - \tilde{S}_s(y)]^2\, dx \quad \rightarrow \quad \text{min.} \tag{82}$$

or

$$\int_{-1}^{y_0} [S(y) - \tilde{S}_s(y)]^2\, dy \quad \rightarrow \quad \text{min.} \tag{83}$$

The minimum condition (82) is fulfilled if (Davis, 1975, Theorem 8.51, p. 171) the coefficients of $\tilde{S}_s(kx+k-1)$ (a finite sum of polynomials $P_r(x)$) are the Fourier coefficients of $S(kx+k-1)$;

$$u_r = \int_{-1}^{1} S(y)\, P_r(x)\, dx \tag{84}$$

Considering equation (77), let

$$\Delta K_M(\cos\psi) = \begin{cases} 0 & , \quad 0 < \psi \leq \psi_0 \\ S(\cos\psi) - \tilde{S}_s(\cos\psi) & , \quad \psi_0 < \psi \leq \pi \end{cases} \tag{85}$$

which may be expanded in a series

$$\Delta K_M(y) = \sum_{n=0}^{\infty} \frac{2n+1}{2}\, Q_{Mn}\, P_n(y) \tag{86}$$

The coefficients are given by

$$Q_{Mn} = \int_{-1}^{y_0} [S(y) - \tilde{S}_s(y)]\, P_n(y)\, dy \tag{87}$$

$$= Q_{1n} - \tilde{Q}_n \tag{88}$$

where

$$\tilde{Q}_n = \int_{-1}^{y_0} \tilde{S}_s(y)\, P_n(y)\, dy \tag{89}$$

and the Q_{1n} are the usual Molodenskii truncation coefficients given by (17). In view of (84) and (80), $\tilde{S}_s(y)$ is the truncated expansion of $S(y)$ in the interval $[-1, y_0]$. The immediate consequence is that

$$Q_{Mn} = 0\,, \qquad 0 \leq n \leq m \tag{90}$$

Indeed, from (87), (80), and (79), and for $n \leq m$, we have

$$\begin{aligned} Q_{1n} - \tilde{Q}_n &= k\int_{-1}^{1} [S(kx+k-1) - \tilde{S}_s(kx+k-1)]\, P_n(kx+k-1)\, dx \\ &= k\int_{-1}^{1}\left(\sum_{r=s+1}^{\infty} \frac{2r+1}{2}\, u_r\, P_r(x)\right) P_n(kx+k-1)\, dx \end{aligned} \tag{91}$$

Now $P_n(kx+k-1)$ is a polynomial in x of degree $n \le m$, which is also expressible as a finite sum of Legendre polynomials $P_s(x)$, $0 \le s \le n$ (see section 5). Therefore $P_n(kx+k-1)$ contains no harmonics above degree m; also $S(kx+k-1) - S_m(kx+x-1)$ has none below degree m (i.e., $r \ge m+1$ in (91)) - the orthogonality of Legendre polynomicals ensures the result (90). The remaining coefficients Q_{Mn} $(n > m)$, characterizing the error (see below), are expected to be relatively small in magnitude, this depending also on the cap radius ψ_0.

The error $\delta N_M'$ (equation (77)) is now more suitably written as

$$\delta N_M' = \frac{R}{2\gamma}\sum_{n=2}^{m} s_n\, \delta(\Delta g_n) + \frac{R}{2\gamma}\sum_{n=m+1}^{\infty} Q_{Mn}\, \Delta g_n \tag{92}$$

The corresponding RMS error is

$$\overline{\delta N_M}' = \frac{R}{2\gamma}\left[\sum_{n=2}^{m} s_n^2\, \delta c_n + \sum_{n=m+1}^{\infty} Q_{Mn}^2\, c_n\right]^{\frac{1}{2}} \tag{93}$$

The second term in (93) arises from the difference between $\tilde{S}_m(y)$ and $S(y)$ for $-1 \le y \le y_0$. However, we cannot choose $\bar{n}$ arbitrarily large, anticipating a reduction of the truncation error beyond significance, unless we have an equally large arsenal of known harmonic coefficients. For example, if we let $\bar{n} > m$ in equation (75), then without difficulty, one can deduce

$$\overline{\delta N_{M1}}' = \frac{R}{2\gamma}\left[\sum_{n=2}^{m} s_n^2\, \delta c_n + \sum_{n=m+1}^{\bar{n}} s_n^2\, c_n + \sum_{n=\bar{n}+1}^{\infty} Q_{Mn}^2\, c_n\right]^{\frac{1}{2}} \tag{94}$$

where $\hat{N}_{M1}$ is given by (76) with $\tilde{S}_m(\cos\psi)$ replaced by $\tilde{S}_{\bar{n}}(\cos\psi)$. Alternately, if $\bar{n} < m$, then it follows just as easily that

$$\hat{N}_{M2} = \frac{R}{4\pi\gamma}\iint_{\sigma_c}[S(\cos\psi) - \tilde{S}_{\bar{n}}(\cos\psi)]\,\Delta g\, d\sigma + \frac{R}{2\gamma}\sum_{n=2}^{\bar{n}} s_n \Delta\hat{g}_n + \frac{R}{2\gamma}\sum_{n=\bar{n}+1}^{m} Q_{Mn}\, \Delta\hat{g}_n \tag{95}$$

with an RMS error

$$\overline{\delta N_{M2}}' = \frac{R}{2\gamma}\left[\sum_{n=2}^{\bar{n}} s_n^2\, \delta c_n + \sum_{n=\bar{n}+1}^{m} Q_{Mn}^2\, \delta c_n + \sum_{n=m+1}^{\infty} Q_{Mn}^2\, c_n\right]^{\frac{1}{2}} \tag{96}$$

The results of a numerical comparison of (93), (94), and (96) are presented in section 5.

Assume, for the moment, that $\delta(\Delta g_n) = 0$, $n = 2, \ldots, m$, so that the application of Schwartz's inequality (Davis, 1975, p. 134) to (77) yields

$$\begin{aligned} |\delta N_M'| &\le \frac{R}{2\gamma}\left(\int_{-1}^{y_0}[S(y) - \tilde{S}_m(y)]^2\, dy\right)^{\frac{1}{2}}\left(\iint_{\sigma-\sigma_c}(\Delta g)^2\, d\sigma\right)^{\frac{1}{2}} \\ &\le \frac{R}{2\gamma}\,\Gamma\left(\int_{-1}^{y_0}[S(y) - \tilde{S}_m(y)]^2 dy\right)^{\frac{1}{2}} \end{aligned} \tag{97}$$

where, for all θ, λ, $\iint_{\sigma-\sigma_c}(\Delta g(\theta', \lambda'))^2\, d\sigma \le \Gamma^2 =$ constant, and where $|\cdot|$

denotes the absolute value. We see that implicit in the attempt to minimize the difference between $\widetilde{S}_*(y)$ and $S(y)$ in the interval $[-1, y_0]$ is the minimization, in some sense, of the error $\delta N_M'$. The original objective to obtain the most rapidly converging error series is inconsequential as far as equation (93) is concerned (if $\delta c_n = 0$), since the error is not characterized solely by the coefficients of the analytic function $\widetilde{S}_*(y)$. Only in (94), where $\bar{n} > m$, do the s_n play a part in the truncation error. Nevertheless, the point must be stressed that an increased convergence rate does not imply a decrease in the error (see section 5). Molodenskii et al. (1962) adopt the approach which starts with equation (77) subject to the condition (83), and no mention of convergence rates needs to enter the discussion.

Interestingly, the error kernel ΔK_M, given by equation (85) is discontinuous at $\psi = \psi_0$! The discontinuity is relatively "small"; nevertheless, Meissl's strategy is immediately called to mind - subtract from the error kernel its value at ψ_0 and thus increase the rate of convergence of the error series. Table 3 shows the subsequent increase in the RMS truncation error ($\delta c_n = 0$, $\bar{n} = m = 20$), as the improved convergence rate is not realized until n is very large ($n > 170$, for $\psi_0 = 10°$, see Table 3a). In a different approach, Hsu Houtze and Zhu Zhuowen (1979) propose that the function $\widetilde{S}_*$ be determined by simultaneously removing the discontinuities of ΔK_M and its first derivative at $\psi = \psi_0$. This they accomplish by subjecting the minimum condition (83), through the use of Lagrange multipliers, to the constraints $S(y_0) = \widetilde{S}_*(y_0)$ and $S'(y_0) = \widetilde{S}_*'(y_0)$. Whether this method finally produces smaller truncation errors is not clear since the difference between $S(y)$ and $\widetilde{S}_*(y)$ in the interval $[-1, y_0]$, as measured by the norm (78), must necessarily increase under the above constraints; and it was shown already (equation (75)) that the truncation error is directly influenced by this difference in the functions.

Table 3. RMS errors in N, assuming errorless potential coefficients to degree m = 20, using Molodenskii's kernel and his kernel modified by removing its discontinuity*. (Errors in meters.)

ψ_0	$\overline{\delta N_M'}$	$\overline{\delta N_{M3}'}$ *
1°	1.93	2.53
2°	1.13	1.74
5°	.28	.47
10°	.03	.05

* using the error kernel $\Delta K_{M3}(y) = \begin{cases} S(y_0) - \tilde{S}_s(y_0), & y_0 \leq y < 1 \\ S(y) - \tilde{S}_s(y), & -1 \leq y < y_0 \end{cases}$

Table 3a. Coefficients Q_M and Q_{M3} of the error kernels ΔK_M and ΔK_{M3} (see Table 3); $\psi_0 = 10°$.

n	Q_M	Q_{M3}
30	-5.98×10^{-4}	-1.28×10^{-3}
100	4.97×10^{-5}	-5.37×10^{-5}
200	-2.50×10^{-5}	-1.15×10^{-5}
300	-2.05×10^{-6}	1.29×10^{-6}
1500	6.00×10^{-7}	-8.11×10^{-6}

5. Computational Procedures and Results

The previous section dealt primarily with Molodenskii's theory on the reduction of the truncation error when the undulation is obtained by integrating Δg over a cap σ_c and from harmonic coefficients. In this section, the theory is implemented and all necessary working formulas are derived. The essential quantities to be determined are the coefficients s_n of the function $\widetilde{S}_{\bar{n}}(y)$, since they are required in both the calculation of $\hat{N}_M$ (equation (76)) and the estimation of the error (equation (93)). The expansions for $\widetilde{S}_{\bar{n}}(y)$ are repeated for convenience:

$$\widetilde{S}_{\bar{n}}(y) = \sum_{n=0}^{\bar{n}} \frac{2n+1}{2} s_n P_n(y) \tag{98}$$

$$\widetilde{S}_{\bar{n}}(y) = \sum_{r=0}^{\bar{n}} \frac{2r+1}{2} u_r P_r(x) \tag{99}$$

where

$$y = kx + k - 1, \quad x = \frac{y-k+1}{k}, \quad k = \cos^2\frac{\psi_0}{2} \tag{100}$$

The Fourier coefficients Q_{1n}, $\widetilde{Q}_n$, s_n, u_r are given by

$$Q_{1n} = \int_{-1}^{y_c} S(y)\, P_n(y)\, dy, \quad n \geq 0 \tag{101}$$

$$\widetilde{Q}_r = \int_{-1}^{y_0} \widetilde{S}_{\bar{n}}(y)\, P_n(y)\, dy, \quad n \geq 0 \tag{102}$$

$$s_n = \int_{-1}^{1} \widetilde{S}_{\bar{n}}(y)\, P_n(y)\, dy, \quad 0 \leq n \leq \bar{n} \tag{103}$$

$$u_r = \int_{-1}^{1} S(y)\, P_r(x)\, dx, \quad 0 \leq r \leq \bar{n} \tag{104}$$

Changing variables from x to y in (104) yields

$$u_r = \frac{1}{k}\int_{-1}^{y_0} S(y)\, P_r\left(\frac{y-k+1}{k}\right) dy; \quad (\psi_0 \neq \pi), \quad 0 \leq r \leq \bar{n} \tag{105}$$

Now, $P_r((y-k+1)/k)$ is a polynomial in y of degree r. Furthermore, it is defined for all y, in particular in the interval $[-1,1]$. Therefore, $P_r((y-k+1)/k)$ can be expanded as a linear combination of the orthogonal (independent) polynomials $P_n(y)$, $0 \leq n \leq r$, which generate the space of all (real) polynomials of degree r defined in $[-1,1]$:

$$P_r\left(\frac{y-k+1}{k}\right) = \sum_{n=0}^{r} \frac{2n+1}{2} h_{rn} P_n(y) \tag{106}$$

where the coefficients h_{rn} are given by

$$h_{rn} = \int_{-1}^{1} P_r\left(\frac{y-k+1}{k}\right) P_n(y)\,dy\ ; \quad r,n \geq 0 \tag{107}$$

We note that by the orthogonality of Legendre polynomials,

$$h_{rn} = 0 \quad \text{if} \quad n > r \tag{108}$$

since $P_r((y-k+1)/k)$ has no components beyond degree r. Substituting (106) into (105) results in

$$u_r = \frac{1}{k}\sum_{n=0}^{r} \frac{2n+1}{2} h_{rn} \int_{-1}^{y_0} S(y)\,P_n(y)\,dy$$

$$= \frac{1}{k}\sum_{n=0}^{r} \frac{2n+1}{2} h_{rn}\, Q_{1n}\,, \quad 0 \leq r \leq n \tag{109}$$

Using the expansion (99) in equation (103), we obtain

$$s_n = \int_{-1}^{1} \sum_{r=0}^{\bar{n}} \frac{2r+1}{2} u_r\, P_r\left(\frac{y-k+1}{k}\right) P_n(y)\,dy$$

$$= \sum_{r=n}^{\bar{n}} \frac{2r+1}{2} u_r\, h_{rn}\,, \quad 0 \leq n \leq \bar{n} \tag{110}$$

where (108) has been used. Finally, from (98) and (102)

$$\tilde{Q}_n = \int_{-1}^{y_0} \sum_{r=0}^{\bar{n}} \frac{2r+1}{2} s_r\, P_r(y)\; P_n(y)\,dy$$

$$= \sum_{r=0}^{\bar{n}} \frac{2r+1}{2} s_r\, e_{rn}\,, \quad n \geq 0 \tag{111}$$

where suitable working formulas for e_{rn} are given by (68) to (70).

The truncation coefficients Q_{1n}, required in the computation of s_n and Q_{M_n}, have been studied extensively, and efficient and accurate recursive formulas exist in the literature. Hagiwara's (1976) elegant solution is used here. A recursive relationship among the h_{rn} is established using equation (31) and several integrations by parts (cf. Appendix B). The final result is

$$\left.\begin{aligned}
h_{r+1,n} &= h_{r-1,n} + \frac{2r+1}{k}\left[\frac{h_{r,n-1}}{2n-1} - \frac{h_{r,n+1}}{2n+3}\right], \quad 0 < n \leq r+1 \\
h_{r,0} &= \frac{k}{2r+1}\left[P_{r+1}\left(\frac{2-k}{k}\right) - P_{r-1}\left(\frac{2-k}{k}\right)\right], \quad r > 0 \\
h_{0,0} &= 2\,; \quad h_{1,0} = \frac{2}{k}(1-k)\,; \quad h_{1,1} = \frac{2}{3}\frac{1}{k} \\
h_{r\,n} &= 0\,, \quad n > r
\end{aligned}\right\} \tag{112}$$

Formula (112) is numerically unstable (at least for $\psi_0 \leq 30^\circ$) and is useless in practice. Instead, we resort to a computationally more burdensome and time consuming closed expression for h_{rn} that was developed by Molodenskii (1958):

$$h_{rn} = (2n+1)\, k^{-r} \sum_{i=0}^{p} \binom{p}{i}\binom{q}{i} (1-k)^{i+1}, \quad 0 \leq n < r$$

$$p = r-n-1, \; q = r+n \tag{113}$$

$$h_{rr} = \frac{2}{2r+1}\, k^{-r}, \quad r \geq 0$$

where, e.g., $\binom{p}{i}$ denotes the binomial coefficient $\frac{p!}{i!\,(p-i)!}$. Some care must be exercised when calculating these coefficients on a digital computer. The computational accuracy of (113) can be tested using equation (106). This has been done by the author to degree $r = 80$ with completely satisfactory results (for $\psi_0 = 10^\circ$, 16-digit accuracy at degree 0 deteriorated to 11-digit accuracy at degree 80).

In Figures 9 and 10 the RMS errors corresponding to Molodenskii's modifications to the error kernel are compared only with Meissl's case. Figures 6 and 7 already indicate strikingly that through a proper modification of the kernel the truncation error can be reduced significantly. Therefore, the question is whether to choose Meissl's almost trivial modification or to adopt the computationally more elaborate method of Molodenskii. Of all the alternatives shown in Figure 9 (based on errorless harmonic coefficients to degree 20 (GEM 9)), Molodenskii's modification (i.e. $\bar{n} = m = 20$) yields the smallest truncation error for most ψ_0. The modifications defined by the condition $\bar{n} \neq m$, as suggested in section 4 (equations (94) and (96), here with $\bar{n} = 25$ and $\bar{n} = 10$, respectively; and $\delta c_n = 0$), although giving truncation errors slightly smaller than $\overline{\delta N_2'}$, show no improvement over the case when $\bar{n} = m = 20$. From Figure 9 ($\delta c_n = 0$) and for $\psi_0 = 10^\circ$, $\overline{\delta N_M'} = .03$ m, $\overline{\delta N_{M1}'} = .09$ m, $\overline{\delta N_{M2}'} = .15$ m, and $\overline{\delta N_2'} = .26$ m. Introducing the errors (standard deviations) of the GEM 9 20-degree reference field (see section 3), but still neglecting integration and gravity data errors, the simple modification of Meissl for $\psi_0 > 6^\circ$ emerges as an effective means to reduce the total RMS error, surpassing even Molodenskii's method (see Figure 10). However, the alternative modification represented by equations (95) and (96), with $\bar{n} = 10$, yields consistently the smallest errors for all ψ_0. For example, at $\psi_0 = 10^\circ$, $\overline{\delta N_{M2}'} = .33$ m, $\overline{\delta N_2'} = .41$ m, $\overline{\delta N_M'} = .46$ m, and $\overline{\delta N_{M1}'} = .54$ m.

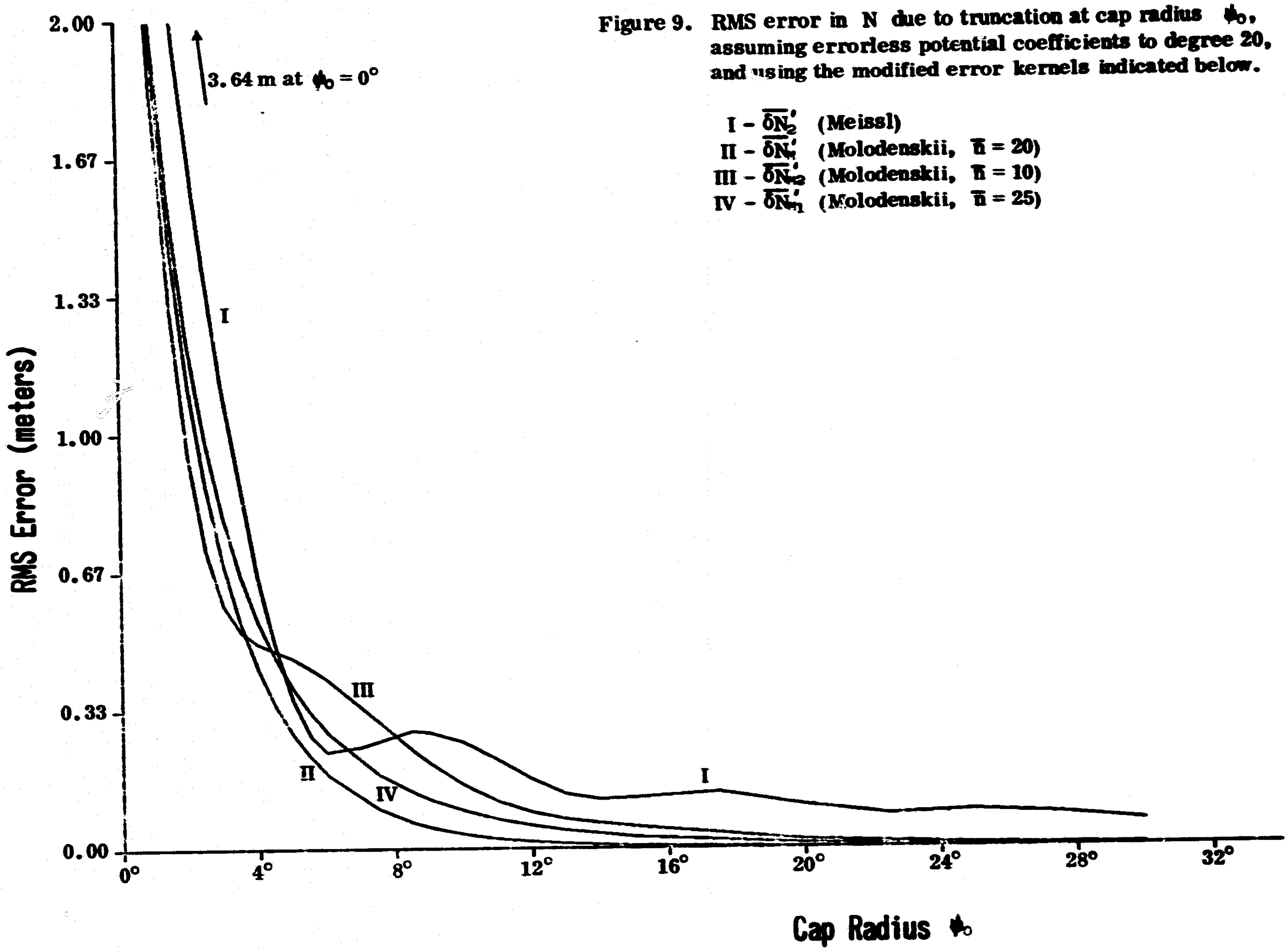


Figure 9. RMS error in N due to truncation at cap radius ψ_0, assuming errorless potential coefficients to degree 20, and using the modified error kernels indicated below.

I - $\overline{\delta N_2'}$ (Meissl)
II - $\overline{\delta N_1'}$ (Molodenskii, $\bar{n} = 20$)
III - $\overline{\delta N_{n2}'}$ (Molodenskii, $\bar{n} = 10$)
IV - $\overline{\delta N_{n1}'}$ (Molodenskii, $\bar{n} = 25$)

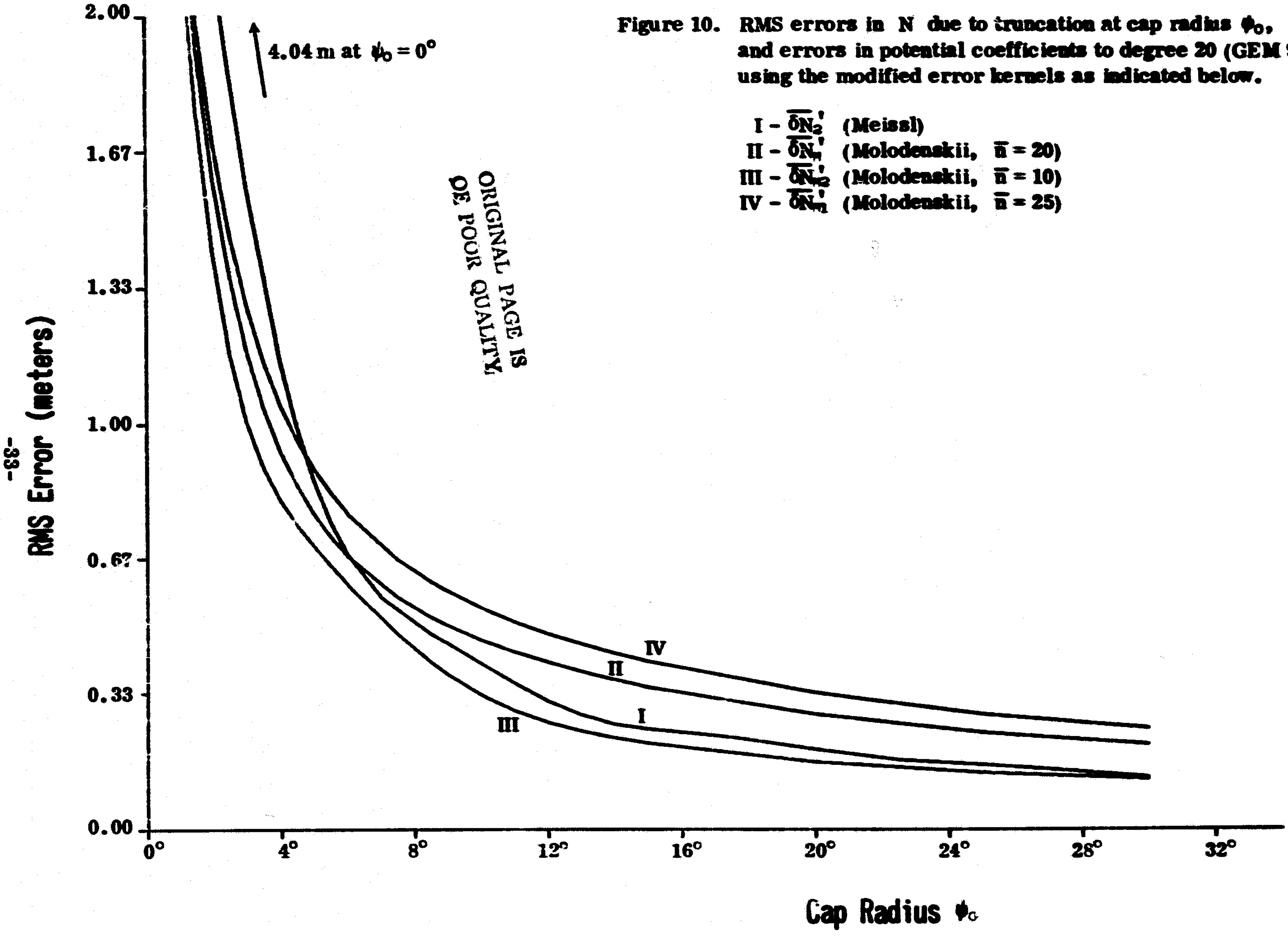


Figure 10. RMS errors in N due to truncation at cap radius ψ_0, and errors in potential coefficients to degree 20 (GEM 9), using the modified error kernels as indicated below.

I - $\overline{\delta N_2}'$ (Meissl)
II - $\overline{\delta N_n}'$ (Molodenskii, $\bar{n} = 20$)
III - $\overline{\delta N_{n2}}'$ (Molodenskii, $\bar{n} = 10$)
IV - $\overline{\delta N_n}'$ (Molodenskii, $\bar{n} = 25$)

6. Numerical Tests with Actual Gravity Data, and the Atmospheric Correction

The computations of the geoid undulation according to the familiar unmodified case (equation (45)), as well as with Meissl's modification (equation (47)) have been carried out with actual gravity data by Rapp (1980) for two oceanic areas, each spanning 30° in both latitude and longitude: Tonga Trench Area and Indian Ocean Area. The (point) undulations were determined on a 1°-grid from an integration of 1° x 1° mean gravity anomalies within a cap of radius $\psi_0 = 10°$ (these were determined from a combination of terrestrial and altimetric anomalies) and from the GEM 9 potential coefficients. Subsequently, these computed undulations were compared to the GEOS-3 altimeter geoid for which Rapp (1980) gives average accuracies of ±0.61 m (Tonga Trench Area) and ±0.71 m (Indian Ocean Area). Table 4 lists the corresponding mean and RMS differences between the computed and altimeter geoids (ibid, p. 6 and p. 9). With the same data as described above, Molodenskii's modification (equation (76)), as well as the alternative method with $\bar{n} = 10$ (equation (95)) were tested similarly; the results are also shown in Table 4. Any of the modified kernels yields a geoid that agrees better with the altimeter geoid by almost 50% than the geoid determined using the unaltered Stokes' function. The RMS differences between the geoids based on the modified kernels and the altimeter geoid approach the quoted accuracy of the latter. It is therefore not possible to rate one modified kernel better than the other from these tests.

The set of gravity anomalies used for the integration, as mentioned above, was derived from both terrestrial anomalies (obtained on the actual geoid) and altimetric anomalies (acquired through collocation from altimetry). In determining the precise relationship between these two types of anomalies, one must heed the effect of the atmosphere (see also Rapp, 1979). The details of this relationship are explained below (cf. Rapp and Rummel, 1975). While these elaborations constitute a digression from the main text, the final results do pertain to the application of truncation theory.

To simplify the discussion, the sea surface is assumed to be a stationary, equipotential surface, which then serves as the geoid, by definition. Moreover, we suppose that the geoid encloses all terrestrial masses, but not the surrounding atmosphere. Accordingly, the gravity potential on the geoid, W_0, is that potential which would actually be observed. The altimeter measurement, being purely geometric, provides directly the geoidal surface as defined above. In this respect, it is necessary to define only the size, shape, and position of the reference ellipsoid. Let the ellipsoid be centered at the geoid's center of mass and aligned with the earth's rotational axis. For subsequent convenience, however, the scale of the ellipsoid will be obtained dynamically be equating the ellipsoidal mass with the geoidal mass plus the mass of the atmosphere and by further defining the normal potential on the ellipsoid, U_0, to equal W_0. Given the ellipsoid's flattening and rotational rate, its semimajor axis is then uniquely determined (see Heiskanen and Moritz, 1967, p. 110).

Before the undulations from altimetry can be processed further into gravity anomalies (via the inverse Stokes' formula, or collocation - in essence, solutions of the boundary-value problem), theory demands that the atmosphere which envelops the geoid be removed. According to Moritz (1974), this is conveniently achieved by redistributing all atmospheric masses (assumed to be spherically symmetric) above the ellipsoid (including fictitious atmospheric masses between the geoid and ellipsoid) inside the geoid, in a manner which leaves the center of mass unchanged. The introduction of mass into the geoid results in a cogeoid (see Heiskanen and Moritz, 1967, p. 141):

$$N^c = N - \Delta N \tag{114}$$

where ΔN is the indirect effect produced by the change in potential δW_A due to this redistribution:

$$\Delta N = \frac{1}{\gamma}\,\delta W_A \tag{115}$$

From (Moritz, 1974, p. 13) we have

$$\delta W_A = -k\int_{r=R}^{\infty}\frac{M(r')\,dr'}{(r')^2} \tag{116}$$

where k is the gravitational constant, and $M(r')$ is the mass of the atmosphere above the sphere of radius r'. δW_A is the difference between the potential due to the atmosphere lying above the geoid and the potential of the atmosphere redistributed below the geoid. This difference is rather small, resulting in an indirect effect, ΔN, of approximately 6 mm (Rummel and Rapp, 1976). The corresponding change in gravity on the geoid is

$$\delta g_A = -\frac{\partial}{\partial r}(\delta W_A)\Big|_{r=R} = -k\frac{M(R)}{R^2} \tag{117}$$

that is, the (negative) attraction of the atmosphere (which is now inside the geoid). Note that (under the simplification of spherical symmetry) the atmosphere, when situated above the geoid, has no effect on the geoidal gravity; and hence, the redistribution affects gravity more substantially (δg_A = -.87 mgal at the geoid, see Moritz (1974)). Moritz has chosen the signs such that the potential and gravity anomaly on the original geoidal surface become, respectively,

$$W^c = W_0 - \delta W_A \tag{118}$$

and

$$\Delta g^c = \Delta g_G - \delta g_A \tag{119}$$

where Δg_G is the geoidal gravity anomaly prior to redistribution of the external masses. Observe that the gravity potential on the cogeoid is W_0 and that the reference ellipsoid has undergone no changes in the process of redistributing the atmospheric masses. From our definitions, the normal potential on the ellipsoid equals the gravity potential on the cogeoid and both surfaces enclose the same mass. Therefore, the cogeoid undulations have no bias (Heiskanen and Moritz, 1967, p. 101).

The indirect effect, ΔN (~6 mm), in practice is evidently neglected. The altimeter undulations may thus be identified as undulations of the cogeoid (with no external masses), having a global average equal to zero, and which therefore are directly usable in collocation to estimate gravity anomalies. However, the resulting anomalies must be correctly interpreted as anomalies on the cogeoid referring to an ellipsoid containing both the terrestrial and atmospheric masses. These altimetric anomalies, Δg_{Alt}, are consistent with Δg^{G} (neglecting the indirect effect of 6 mm on the value of gravity). The terrestrial anomalies, Δg_{Ter}, on the other hand, are obtained on the actual geoid (with no internal atmospheric masses), but also refer to an ellipsoid containing the atmosphere (assume it is the reference ellipsoid defined above). The two types of anomalies Δg_{Ter} and Δg_{Alt} are rendered compatible, according to (119), by either subtracting the atmospheric correction δg_A from Δg_{Ter} :

$$\Delta g^{G} = \Delta g_{Ter} - \delta g_A \rightarrow \text{ consistent with } \Delta g_{Alt} \tag{120}$$

or adding δg_A to the altimetric anomalies:

$$\Delta g_G = \Delta g_{Alt} + \delta g_A \rightarrow \text{ consistent with } \Delta g_{Ter} \tag{121}$$

At sea level, the correction δg_A is a constant. Hence, for convenience, the set of gravity anomalies which enter the truncated Stokes' integral was obtained by merging anomalies Δg_G (equation (121)) and Δg_{Ter}, and subsequently corrected by $-\delta g_A$ so that the integral is validly applied. Regarding the integration, this constant part can be treated separately.

Substituting (119) into any of the equations for the undulation (45), (47), (57), (76), or (95), we have

$$\hat{N} = \frac{R}{4\pi\gamma} \iint_{\sigma_C} K(\cos\psi)\, \Delta g_G \, d\sigma + \hat{N}' - \frac{R}{4\pi\gamma}\, \delta g_A \iint_{\sigma_C} K(\cos\psi)\, d\sigma \tag{122}$$

where $\hat{N}'$ represents the contribution to $\hat{N}$ from the harmonic coefficients, which are already based on a geoid (cogeoid) that contains the mass of the atmosphere. Let the last term in (122) be denoted by δN_A. This is the correction to be applied to the computed integral (the first term). If the kernel K is the usual Stokes' function S, then

$$\begin{aligned}\delta N_{1A} &= \frac{-R}{2\gamma}\delta g_A \int_{y_C}^{1} S(y)\,dy = \frac{-R}{2\gamma}\delta g_A \left[\int_{-1}^{1} S(y)\,dy - \int_{-1}^{y_0} S(y)\,dy\right] = \frac{-R}{2\gamma}\delta g_A\,(0 - Q_{1,0}) \\ &= \frac{R}{2\gamma}\delta g_A\, Q_{1,0}\end{aligned} \tag{123}$$

Similarly, for $K_2(y) = S(y) - S_0$, using (41)

$$\delta N_{2A} = \frac{-R}{2\gamma}\,\delta g_A \left[\int_{-1}^{1} (S(y) - S_0)\,dy - (Q_{2,0} - 2S_0)\right] = \frac{R}{2\gamma}\delta g_A\, Q_{2,0} \tag{124}$$

And for $K_M(y) = S(y) - \tilde{S}_{\bar{n}}(y)$, we obtain with (90)

$$\delta N_{MA} = \frac{-R}{2\gamma}\,\delta g_A \left[\int_{-1}^{1} (S(y) - \tilde{S}_{\bar{n}}(y))\,dy - Q_{M,0}\right] = \frac{-R}{2\gamma}\,\delta g_A\,(0 - s_0 - 0)$$

$$= \frac{R}{2\gamma}\,\delta g_A\, s_0 \tag{125}$$

For $\psi_0 = 10^\circ$, $\delta N_{1A} = 1.17$ m, $\delta N_{2A} = 0.57$ m, $\delta N_{MA}|_{\bar{n}=20} = 0.43$ m, and $\delta N_{MA}|_{\bar{n}=10} = 0.60$ m. These values should be added to $(R/4\pi\gamma)\iint_\sigma K(\cos\psi)\Delta g_G\,d\sigma + N'$ (equation (122)) to obtain the cogeoidal (or geoidal, since the indirect effect is negligible) undulation $\hat{N}$.

Table 4. Computed Geoid undulations ($\psi_0 = 10^\circ$, m = 20) minus GEOS-3 Altimeter Undulations: Mean and RMS Differences in meters.

Stokes' function	Tonga Trench Area		Indian Ocean Area	
	Mean Difference	RMS Difference	Mean Difference	RMS Difference
Unmodified	0.8	2.2	0.8	1.6
Meissl's Mod.	0.4	1.1	0.2	0.7
Molodenskii's mod. $\bar{n}$ = m = 20	0.3	0.9	0.1	0.6
Molodenskii's mod. $\bar{n}$ = 10, m = 20	0.4	1.1	0.3	0.7

7. Colombo's Method

This section is devoted to the method developed by Colombo (1977), whose essential strategy is not to modify the error kernel in order to speed up the convergence rate of the error series, but to simply determine a kernel for the integration over the cap which minimizes the truncation error. Under the condition of a "band-limited" gravity field (that is, one with a finite nmuber of harmonic components), the problem thus posed reduces to the solution of a finite number of unknown parameters from an equal number of linear equations. The (square) matrix to be inverted is extremely ill-conditioned, especially for the cap sizes of interest ($\psi_0 \leq 30°$); nevertheless, Colombo (1977) achieved the desired result of significantly smaller truncation errors through a regularization of this matrix. In the following elaborations, analytic (recursive) expressions are derived for the decomposition of the matrix into lower and upper triangular matrices, leading further to analytic (recursive) expressions for the unknown quantities to be solved. The near singularity of the system is thereby not eliminated, but it will be shown that rigorously, without any form of regularization, the problem as stated above cannot be solved satisfactorily. In addition, if the gravity field is not "band-limited", then the error (without regularization) is far from minimal.

The undulation is decomposed as follows

$$\begin{aligned} N &= \frac{R}{4\pi\gamma} \iint_\sigma S(\cos\psi)\,\Delta g\, d\sigma \\ &= \frac{R}{4\pi\gamma} \iint_{\sigma_c} K_c(\cos\psi)\,\Delta g\, d\sigma + \frac{R}{4\pi\gamma} \iint_\sigma \Delta K_c(\cos\psi)\,\Delta g\, d\sigma \end{aligned} \tag{126}$$

where by definition (the reason for this choice will be apparent below)

$$K_c(y) = \sum_{n=0}^{\bar{n}} \frac{2n+1}{2}\, v_n\, P_n(y) \tag{127}$$

so that, evidently, we must have

$$\Delta K_c(\cos\psi) = \begin{cases} S(\cos\psi) - K_c(\cos\psi)\,, & 0 < \psi \leq \psi_0 \\ S(\cos\psi)\,, & \psi_0 < \psi \leq \pi \end{cases} \tag{128}$$

The unknown coefficients v_n are to be determined under the requirement of a minimum error. This error is given by the second integral in (126), which in view of (128) becomes

$$\delta N_c = \frac{R}{4\pi\gamma} \iint_\sigma S(\cos\psi)\,\Delta g\, d\sigma - \frac{R}{4\pi\gamma} \iint_{\sigma_c} K_c(\cos\psi)\,\Delta g\, d\sigma \tag{129}$$

$$= \frac{R}{2\gamma} \sum_{n=2}^{\infty} t_n\, \Delta g_n - \frac{R}{2\gamma} \sum_{n=2}^{\infty} q_n\, \Delta g_n \tag{130}$$

The coefficients $2\pi t_n$ are the eigenvalues of the first integral operator in (129) (see section 2), while the q_n are the Fourier coefficients of the kernel

$$\overline{K}_c(\cos\psi) = \begin{cases} K_c(\cos\psi), & 0 < \psi \leq \psi_0 \\ 0, & \psi_c < \psi \leq \pi \end{cases} \tag{131}$$

that is,

$$q_n = \int_{y_0}^{1} K_c(y)\, P_n(y)\, dy, \qquad 0 \leq n \leq \bar{n} \tag{132}$$

Then clearly the eigenvalues of the second integral operator in (129) are $2\pi q_n$. The result (130) is finally established by recalling our assumptions that $\Delta g_0 = 0 = \Delta g_1$. More compactly, we have

$$\delta N_c = \frac{R}{2\gamma} \sum_{n=2}^{\infty} (t_n - q_n)\, \Delta g_n \tag{133}$$

Considering (127) and (132), the coefficients q_n are functions of the parameters v_n:

$$q_n = \sum_{r=0}^{\bar{n}} \frac{2r+1}{2}\, v_r \int_{y_0}^{1} P_r(y)\, P_n(y)\, dy, \qquad 0 \leq n \leq \bar{n} \tag{134}$$

For the moment, assume that the gravity field is composed of a finite number of harmonic components; i.e. there exists an integer $\bar{n}$ such that

$$\Delta g_n = 0, \qquad n > \bar{n} \tag{135}$$

This bounding degree $\bar{n}$ is also used in the definition of K_c. Then from (133)

$$\delta N_{c\bar{n}} = \frac{R}{2\gamma} \sum_{n=2}^{\bar{n}} (t_n - q_n)\, \Delta g_n \tag{136}$$

This is identically zero (the smallest possible error) if

$$t_n = q_n, \qquad 0 \leq n \leq \bar{n} \tag{137}$$

which with (134) becomes

$$t_n = \sum_{r=0}^{\bar{n}} \frac{2r+1}{2}\, v_r \int_{y_0}^{1} P_r(y)\, P_n(y)\, dy, \qquad 0 \leq n \leq \bar{n} \tag{138}$$

The (known) quantities t_n are given by (4). In matrix notation (138) is

$$T = A\, V \tag{139}$$

where the positive definite matrix A (see Colombo (1977)) has elements

$$a_{rn} = \frac{2n+1}{2}\int_{y_0}^{1} P_r(y)\, P_n(y)\, dy\,; \quad r,n = 0,\ldots,\bar{n} \tag{140}$$

and the vectors T, V contain elements t_n $(n = 0,\ldots,\bar{n})$ and v_r $(r = 0,\ldots,\bar{n})$, respectively. Theoretically, under the condition (135), the system (139) of $\bar{n}+1$ equations can be solved for the $\bar{n}+1$ parameters v_n, yielding a kernel K_c for which the truncation error $\delta N_{c\bar{n}}$ is zero. The gravity field in reality is described by an infinity of harmonics, but this only prevents the error from being absolutely zero. It is hoped, if (135) is approximately true (say, if $\bar{n} = 180$), that the actual erorr

$$\delta N_c = \frac{R}{2\gamma}\sum_{n=\bar{n}+1}^{\infty} (t_n - q_n)\, \Delta g_n \tag{141}$$

remains small.

Proceeding with the derivation of a recursive formula for the coefficients v_n, we note that from (127)

$$v_n = \int_{-1}^{1} K_c(y)\, P_n(y)\, dy \tag{142}$$

K_c can also be expanded in the interval $[y_0, 1]$, where $y_0 = \cos\psi_0$; this requires a corresponding orthogonal basis. Let

$$y = \ell z - \ell + 1\,, \quad \text{where} \quad \ell = \sin^2\frac{\psi_0}{2} \tag{143}$$

then

$$\int_{-1}^{1} P_s(z)\, P_t(z)\, dz = \frac{1}{\ell}\int_{y_0}^{1} P_s\!\left(\frac{y+\ell-1}{\ell}\right) P_t\!\left(\frac{y+\ell-1}{\ell}\right) dy = \begin{cases} 2/(2s+1) & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases} \tag{144}$$

That is, the functions $P_n((y+\ell-1)/\ell)$ form an orthogonal basis for polynomials defined in $[y_0, 1]$. Therefore, the expansion of K_c in $[y_0, 1]$ is

$$K_c(y) = K_c(\ell z - \ell + 1) = \sum_{n=0}^{\bar{n}} \frac{2n+1}{2}\, w_n\, P_n(z) \tag{145}$$

This sum is finite since K_c is a polynomial of degree $\bar{n}$ in y, and hence in z. The Fourier coefficients are

$$w_n = \int_{-1}^{1} K_c(\ell z - \ell + 1)\, P_n(z)\, dz\,, \quad 0 \le n \le \bar{n} \tag{146}$$

which upon inserting (127) become

$$w_n = \int_{-1}^{1} \sum_{r=0}^{\bar{n}} \frac{2r+1}{2}\, v_r\, P_r(\ell z - \ell + 1)\, P_n(z)\, dz$$

$$= \sum_{r=0}^{\bar{n}} \frac{2r+1}{2}\, g_{rn}\, v_r\,, \quad 0 \le n \le \bar{n} \tag{147}$$

where

$$g_{rn} = \int_{-1}^{1} P_r(\ell z - \ell + 1)\, P_n(z)\, dz \; ; \quad r,n = 0,\ldots,\overline{n} \tag{148}$$

Since $P_r(\ell z - \ell + 1)$, when expanded in terms of $P_n(z)$, has no components beyond degree r, the orthogonality of Legendre polynomials guarantees that (cf. equation (108))

$$g_{rn} = 0 \, , \quad n > r \tag{149}$$

Hence

$$w_n = \sum_{r=n}^{\overline{n}} \frac{2r+1}{2} g_{rn} v_r \, , \quad 0 \le n \le \overline{n} \tag{150}$$

If we stipulate that $\delta N_{c\overline{n}} = 0$, then from (137) and (132)

$$t_n = \int_{y_0}^{1} K_c(y)\, P_n(y)\, dy$$

$$= \ell \int_{-1}^{1} K_c(\ell z - \ell + 1)\, P_n(\ell z - \ell + 1)\, dz \tag{151}$$

Substituting (145) yields

$$t_n = \ell \int_{-1}^{1} \sum_{r=0}^{\overline{n}} \frac{2r+1}{2} w_r\, P_r(z)\, P_n(\ell z - \ell + 1)\, dz$$

$$= \ell \sum_{r=0}^{n} \frac{2r+1}{2} g_{nr}\, w_r \, , \quad 0 \le n \le \overline{n} \tag{152}$$

where (149) has been used. Let W be a vector of dimension $(\overline{n}+1)$ containing the coefficients w_r, and further let the $(\overline{n}+1) \times (\overline{n}+1)$ triangular matrices L and U be defined by

$$L = \ell \begin{pmatrix} \frac{1}{2} g_{0,0} & 0 & \cdots\cdots & 0 \\ \frac{1}{2} g_{1,0} & \frac{3}{2} g_{1,1} & \cdots\cdots & 0 \\ \vdots & & \ddots & \vdots \\ \frac{1}{2} g_{\overline{n},0} & \frac{3}{2} g_{\overline{n},1} & \cdots\cdots & \frac{2\overline{n}+1}{2} g_{\overline{n},\overline{n}} \end{pmatrix} \tag{153}$$

$$U = \begin{pmatrix} \frac{1}{2} g_{0,0} & \frac{3}{2} g_{1,0} & \cdots\cdots & \frac{2\overline{n}+1}{2} g_{\overline{n},0} \\ 0 & \frac{3}{2} g_{1,1} & \cdots\cdots & \frac{2\overline{n}+1}{2} g_{\overline{n},1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots\cdots & \frac{2\overline{n}+1}{2} g_{\overline{n},\overline{n}} \end{pmatrix} \tag{154}$$

In matrix notation, equations (152) and (150) can then be written concisely as

$$T = L W, \quad W = U V \quad \text{or} \quad T = L U V \tag{155}$$

With equation (139), we arrive at the desired result

$$A = L\,U \tag{156}$$

thus achieving a decomposition of matrix A into lower and upper triangular matrices.

To determine the kernel K_c it is sufficient to solve only for W (see equation (145)). Inverting equation (152), it is easily verified that

$$\left.\begin{aligned} w_r &= \left(\frac{2r+1}{2}\,\ell\, g_{rr}\right)^{-1}\left(t_r - \ell\sum_{n=0}^{r-1}\frac{2n+1}{2}\, g_{rn}\, w_n\right), \quad 0 < r \leq \bar{n} \\ w_0 &= \ell^{-1}\, t_0 \end{aligned}\right\} \tag{157}$$

Therefore, given values for t_r, $r = 0, \ldots, \bar{n}$ (equation (4)) and a method to accurately compute g_{rn}, equations (157) furnish (recursively) all coefficients w_n. The following closed expression for g_{rn} (cf. equation (113)) can be proved, e.g., by mathematical induction:

$$g_{rn} = (-1)^{q-1}\, 2\,\ell^n \sum_{i=0}^{p}\frac{1}{i+1}\binom{p}{i}\binom{q+i}{i}(\ell-1)^{i+1}; \quad 0 \leq n < r \tag{158}$$

$$p = r-n-1\,, \quad q = r+n+1$$

$$g_{rr} = \frac{2\ell^r}{2r+1}, \qquad r \geq 0 \tag{159}$$

However, because $(\ell - 1)^{i+1}$ is close to unity in magnitude (for the cap radii of interest; e.g. $\psi_0 = 10^\circ \rightarrow \ell \approx .0076$) and oscillates in sign, the summation in (158) on a digital computer is associated with a considerable loss in significant digits. A more suitable method to compute the g_{rn} in this case utilizes a recursion formula which is derived in Appendix B:

$$\left.\begin{aligned} g_{r+1,n} &= g_{r-1,n} + \ell\,\frac{2r+1}{2n+1}\,(g_{r,n-1} - g_{r,n+1})\,, \quad 0 < n \leq r+1 \\ g_{r,0} &= \frac{1}{\ell\,(2r+1)}\,(P_{r-1}(y_0) - P_{r+1}(y_0))\,, \quad r > 0 \\ g_{0,0} &= 2\,; \quad g_{1,0} = 2(1-\ell)\,; \quad g_{1,1} = \tfrac{2}{3}\,\ell \\ g_{rn} &= 0\,, \quad n > r \end{aligned}\right\} \tag{160}$$

Since $\ell << 1$, g_{rr} (equation (159)) is an extremely small quantity for large r, while the values of $g_{r,0}$ are quite stable in magnitude as r increases; the values of g_{rn} vary between these bounds for $0 < n < r$. Consequently, the coefficients w_n according to (157) are difficult to compute, being themselves large in magnitude and oscillatory in sign. A considerable moderation in the extreme range of values of g_{rn} is effected by introducing a scaling factor ℓ^{-n}. Let

$$\bar{g}_{rn} = \ell^{-n}\, g_{rn}\,; \quad r,n \geq 0 \tag{161}$$

Then the recursive relationship (160) transforms into

$$\left.\begin{aligned}
\bar{g}_{r+1,n} &= \bar{g}_{r-1,n} + \frac{2r+1}{2n+1}\left(\bar{g}_{r,n-1} - \ell^2 \bar{g}_{r,n+1}\right), \quad 0 < n \le r+1 \\
\bar{g}_{r,0} &= \frac{1}{\ell(2r+1)}\left(P_{r-1}(y_0) - P_{r+1}(y_0)\right), \quad r > 0 \\
\bar{g}_{0,0} &= 2\,; \quad \bar{g}_{1,0} = 2(1-\ell)\,; \quad \bar{g}_{1,1} = \tfrac{2}{3} \\
\bar{g}_{r,n} &= 0, \quad n > r
\end{aligned}\right\} \quad (162)$$

Now (159) becomes $\bar{g}_{rr} = 2/(2r+1)$, $r \ge 0$, and with

$$\bar{w}_n = \ell^n w_n, \quad n \ge 0 \tag{163}$$

equation (157) changes simply to

$$\begin{aligned}
\bar{w}_r &= \ell^{-1} t_r - \sum_{n=0}^{r-1} \frac{2n+1}{2} \bar{g}_{rn} \bar{w}_n, \quad 0 < r < \bar{n} \\
\bar{w}_0 &= \ell^{-1} t_0
\end{aligned} \tag{164}$$

The scaling described above does not alleviate totally the numerical difficulties since the values of $\bar{g}_{rn}$ still vary over a broad range of magnitudes. For example, if $\psi_0 = 10°$, $\bar{n} = 36$, then

$$\bar{g}_{36,0} = -0.119\,; \quad \bar{g}_{36,12} = 1.82 \times 10^{12}\,; \quad \bar{g}_{36,24} = 5.11 \times 10^{10}\,; \quad \bar{g}_{36,36} = 0.0274$$

and $\quad \bar{w}_3 = -1.17 \times 10^3$; $\bar{w}_{13} = -4.20 \times 10^8$; $\bar{w}_{24} = 8.02 \times 10^{14}$; $\bar{w}_{36} = 7.51 \times 10^{21}$

The kernel K_C (equation (145)), in turn, acquires enormous oscillations, a characteristic that is both undesirable and detrimental to the actual truncation error. This error (equation (141)), as well as an indication of the computational accuracy of equations (162) and (164) are both provided through the evaluation of the coefficients q_n. Substituting (143) and (145) into (142) results in

$$\begin{aligned}
q_n &= \ell \int_{-1}^{1} \sum_{r=0}^{\bar{n}} \frac{2r+1}{2} w_r P_r(z) P_n(\ell z - \ell + 1)\, dz \\
&= \ell \sum_{r=0}^{\bar{n}} \frac{2r+1}{2} w_r g_{nr}, \quad n \ge 0
\end{aligned} \tag{165}$$

Or, considering (161), (163), and (149)

$$q_n = \ell \sum_{r=0}^{M} \frac{2r+1}{2} \bar{g}_{nr} \bar{w}_r, \quad n \ge 0 \tag{166}$$

where $M = \min(n, \bar{n})$. For $0 \le n \le \bar{n}$, we must have $t_n = q_n$, where t_n is given by equation (4). Using the IBM System 370, model 168 with an AMDAHL 470 V16-II processor at The Ohio State University, Columbus, Ohio, with extended precision (more than 30 significant digits), the differences $q_n - t_n$ were obtained with $\psi_0 = 10°$,

$\bar{n} = 36$. Equal to zero for $n = 0$, these differences deteriorated progressively until $q_{36} - t_{36} = 2.3 \times 10^{-8}$; but then $q_{37} - t_{37} \approx 2.2 \times 10^{20}$. The latter is essentially the first term in the truncation error. Therefore, while equations (137) are satisfied with at least seven digits of accuracy, the resulting strong oscillations of the kernel K_c are clearly reflected in the power spectrum of the error kernel, that is, the RMS truncation error, thus apparently ruling out any application of this method without some type of regularization of its ill-conditioned character.

The determination of the coefficients v_n is numerically even more unpleasant, since from (150) and (159) we have

$$v_n = \ell^{-n} \left[w_n - \sum_{r=n+1}^{\bar{n}} \frac{2r+1}{2} g_{rn} v_r \right]$$

$$= \ell^{-n} w_n - \sum_{r=n+1}^{\bar{n}} \frac{2r+1}{2} \bar{g}_{rn} v_r \,, \qquad 0 \le n \le \bar{n} - 1 \tag{167}$$

with $\qquad v_{\bar{n}} = \ell^{-\bar{n}} w_{\bar{n}}$

Hence, these coefficients are excessively large ($\psi_0 = 10°$, $\bar{n} = 5 \rightarrow |v_n|$ is on the order of 10^{25} to 10^{28}).

The rigorous solution for the coefficients v_n (or w_n) as described above does not take into consideration the actual error δN_c (equation (141)) once the v_n are determined. Consider the RMS value of δN_c (equation (133)):

$$\overline{\delta N_c} = \frac{R}{2\gamma} \left[\sum_{n=0}^{\infty} (t_n - q_n)^2 c_n \right]^{\frac{1}{2}} \tag{168}$$

This quantity, which is to be minimized, resembles the square root of the sum of squared and weighted residuals in a typical least squares solution. To make the analogy strictly correct, the series in (168) is terminated at degree $\bar{m} > \bar{n}$. Then the coefficients $t_0 = t_1 = 0$, $t_n = 2/(n-1)$, $2 \le n \le \bar{m}$ serve as the observations; the coefficients q_n, $0 \le n \le \bar{m}$ constitute the mathematical model ("adjusted observations"), being linear functions of the parameters v_n, $0 \le n \le \bar{n}$. The degree variances c_n may be interpreted as weights. Note that the case $\bar{m} = \bar{n}$ was discussed above. The degree of variability which enters the solution by allowing $\bar{m} > \bar{n}$ may, through the minimization condition, have a mitigating influence on the magnitudes of the unknown parameters v_n. That is, the system of equations for the case $\bar{m} = \bar{n}$ is known to be unstable, and the introduction of additional equations could have a regularizing effect on the system, since it is then overdetermined. However, the results of several computational experiments with $\bar{m} = 300$, $\bar{n} = 50$ seem to indicate that the near singularity of the original system is not avertible with this approach.

8. Summary and Conclusion

The geoid undulation can be derived from a combination of gravity anomalies in a spherical cap (truncated Stokes' integral) and a (finite) set of harmonic coefficients. The resulting truncation error depends strongly on the kernel that is used for the integration. Several modifications to the kernel, as proposed by various authors, are scrutinized numerically in this report. These modifications are primarily designed to accelerate the convergence of the series which represents the error, thereby attempting to reduce its magnitude. It is found that these methods can claim such a reduction only if a number of low-degree harmonics (whose magnitudes are not directly controlled by the modifications of the kernel) can be deleted from the error and incorporated in the computation of the undulation. This is not an absolute requirement since the error is also a function of the cap radius ψ_0 (see Figure 5); however, for small caps ($5^\circ \leq \psi_0 \leq 30^\circ$), this seems to be the general rule (Figures 6, 7, 9, 10).

The various numerical investigations amply demonstrate that the integration of gravity anomalies in a cap of radius as small as 5° or 10°, supplemented by potential coefficients (e.g. GEM 9), can yield geoidal accuracies below the 1 meter level. For example, in Figure 10 (where integration and anomaly errors are neglected, but errors in GEM 9 are included), Meissl's (1971b) simple modification of the kernel gives an RMS error of 0.41 m for $\psi_0 = 10^\circ$; while a variation of Molodenskii's method ($\bar{n} = 10$) produces an RMS error of 0.33 m ($\psi_0 = 10^\circ$). Geoid undulations, computed from a combination of actual gravity data in caps with $\psi_0 = 10^\circ$ and the GEM 9 harmonic coefficients, have been compared to the corresponding GEOS-3 altimeter undulations which, in the areas considered, have an average accuracy of better than 1 meter. While not directly verifying the results of Figure 10, this comparison does substantiate the significant (50%) improvement over the conventional method that can be achieved by suitable modifications of Stokes' function. It should be noted that the RMS errors of Figures 5, 6, 7, 9, and 10 are average global estimates that are based on the Tscherning and Rapp (1974) gravity model and a spherical approximation of the geoid, the latter being no longer valid at the 20 cm to 30 cm level of accuracy. Furthermore, the errors of the gravity data, as well as errors associated with the numerical integration have been neglected entirely.

Colombo's (1977) method of treating the truncation of Stokes' integral is included to show a different avenue of approach. Some numerical tests have been conducted, but this method requires further development and analysis.

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Appendix A

If $\quad e_{rr} = \int_{-1}^{y_0} P_r^2(y)\,dy\,, \quad r \geq 0\,,$ then

$$(2r+1)\,e_{rr} = (2r-1)\,e_{r-1,r-1} + y_0[P_r^2(y_0)+P_{r-1}^2(y_0)] - 2P_r(y_0)P_{r-1}(y_0),\quad r>0 \qquad (A.1)$$

$$e_{0,0} = 1 + y_0$$

Proof:

The case $r = 0$ is easily verified by noting that $P_0(y) = 1$, for all y. Suppose that $r > 0$, then with (30) we have

$$r\,e_{rr} = \int_{-1}^{y_0} P_r(y)\; r\,P_r(y)\,dy = \int_{-1}^{y_0} P_r(y)\,[y\,P_r'(y) - P_{r-1}'(y)]\,dy \qquad (A.2)$$

Let
$$a_r = \int_{-1}^{y_0} P_r(y)\; y\; P_r'(y)\,dy\,, \qquad r \geq 0 \qquad (A.3)$$

An integration by parts yields

$$a_r = y\,P_r^2(y)\Big|_{-1}^{y_0} - \int_{-1}^{y_0} P_r(y)\,[y\,P_r'(y) + P_r(y)]\,dy$$

Hence
$$2a_r = y_0\,P_r^2(y_0) + P_r^2(-1) - e_{rr} \qquad (A.4)$$

Putting (A.3) into (A.2), we obtain

$$r\,e_{rr} = a_r - \int_{-1}^{y_0} P_r(y)\; P_{r-1}'(y)\,dy \qquad (A.5)$$

Also
$$\begin{aligned}(r+1)e_{r+1,r+1} &= a_{r+1} - \int_{-1}^{y_0} P_{r+1}(y)\; P_r'(y)\,dy \\ &= a_{r+1} - P_{r+1}(y)\;P_r(y)\Big|_{-1}^{y_0} + \int_{-1}^{y_0} P_r(y)\;P_{r+1}'(y)\,dy \qquad (A.6)\end{aligned}$$

The last step follows from an integration by parts. Now add equations (A.5) and (A.6), and substitute (31):

$$r\,e_{rr} + (r+1)\,e_{r+1,r+1} = a_r + a_{r+1} - P_{r+1}(y_0)\,P_r(y_0) + P_{r+1}(-1)\,P_r(-1) + (2r+1)\int_{-1}^{y_0} P_r^2(y)\,dy$$

With (A.4) and $P_r(-1) = (-1)^r$, this becomes

$$\begin{aligned}(r+1+\tfrac{1}{2})\,e_{r+1,r+1} = (-r-\tfrac{1}{2}+2r+1)\,e_{rr} + \tfrac{1}{2}y_0\,P_r^2(y_0) + \tfrac{1}{2} + \tfrac{1}{2}y_0 P_{r+1}^2(y_0) + \tfrac{1}{2} + \\ - P_{r+1}(y_0)\;P_r(y_0) - 1\end{aligned}$$

which simplifies to

$$\frac{2r+3}{2} e_{r+1,r+1} = \frac{2r+1}{2} e_{rr} + \tfrac{1}{2} y_0 [P_r^2(y_0) + P_{r+1}^2(y_0)] - P_r(y_0) P_{r+1}(y_0)$$

Equation (A.1) follows immediately.

Appendix B

If $\quad g_{rn} = \int_{-1}^{1} P_r(\ell z - \ell + 1)\, P_n(z)\, dz\,, \quad r,n \geq 0 \qquad$ (B.1)

then $\quad g_{r+1,n} = g_{r-1,n} + \ell \dfrac{2r+1}{2n+1}(g_{r,n-1} - g_{r,n+1}), \quad \begin{cases} 0 < n \leq r+1 \\ r > 0 \end{cases} \qquad$ (B.2)

$$g_{r,0} = \frac{1}{\ell(2r+1)}[P_{r-1}(y_0) - P_{r+1}(y_0)]\,, \quad r > 0 \tag{B.3}$$

$$g_{0,0} = 2\,; \quad g_{1,0} = 2(1-\ell)\,; \quad g_{1,1} = \tfrac{2}{3}\ell \tag{B.4}$$

$$g_{r,n} = 0\,, \quad n > r \tag{B.5}$$

Proof:

(B.5) has already been established in section 7, and equations (B.4) are easily verified. Using $y = \ell z - \ell + 1$ and (31)

$$g_{r,0} = \frac{1}{\ell}\int_{y_0}^{1} P_r(y)\, dy = \frac{1}{\ell(2r+1)}[P_{r-1}(y_0) - P_{r+1}(y_0)]\,, \quad r > 0$$

thus proving (B.3).

Consider the integral $\int_{-1}^{1} P_r(y)\, P_n'(z)\, dz$. Integrating by parts, we obtain

$$\int_{-1}^{1} P_{r+1}(y)\, P_n'(z)\, dz = 1 - P_{r+1}(-2\ell+1)(-1)^n - \ell\int_{-1}^{1} P_{r+1}'(y)\, P_n(z)\, dz \tag{B.6}$$

$$\int_{-1}^{1} P_{r-1}(y)\, P_n'(z)\, dz = 1 - P_{r-1}(-2\ell+1)(-1)^n - \ell\int_{-1}^{1} P_{r-1}'(y)\, P_n(z)\, dz \tag{B.7}$$

where the primes always denote differentiation with respect to the argument of the function. Subtracting (B.7) from (B.6) and substituting (31) gives

$$\int_{-1}^{1} [P_{r+1}(y) - P_{r-1}(y)]P_n'(z)\, dz = -P_{r+1}(-2\ell+1)(-1)^n + P_{r-1}(-2\ell+1)(-1)^n - (2r+1)\,\ell g_{rn}. \tag{B.8}$$

Also, if (31) is substituted into (B.1), then

$$g_{r+1,n} = \frac{1}{2n+1}\int_{-1}^{1} P_{r+1}(y)\,[P'_{n+1}(z) - P'_{n-1}(z)]\,dz \tag{B.9}$$

$$g_{r-1,n} = \frac{1}{2n+1}\int_{-1}^{1} P_{r-1}(y)\,[P'_{n+1}(z) - P'_{n-1}(z)]\,dz \tag{B.10}$$

Subtract (B.10) from (B.9) and insert (B.8), then

$$\begin{aligned} g_{r+1,n} - g_{r-1,n} &= \frac{1}{2n+1}\int_{-1}^{1}[P_{r+1}(y) - P_{r-1}(y)][P'_{n+1}(z) - P'_{n-1}(z)]\,dz \\ &= \frac{1}{2n+1}[-P_{r+1}(-2\ell+1)(-1)^{n+1} + P_{r-1}(-2\ell+1)(-1)^{n+1} - (2r+1)\,\ell g_{r,n+1} + \\ &\quad - (-P(-2\ell+1)(-1)^{n-1} + P_{r-1}(-2\ell+1)(-1)^{n-1} - (2r+1)\,\ell g_{r,n-1})] \\ &= \frac{\ell(2r+1)}{2n+1}[g_{r,n-1} - g_{r,n+1}] \end{aligned}$$

thus proving also (B.2).